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MARKOV CHAIN ANALYSIS OF A SITUATION WHERE CANNIBALIZATION IS THE ONLY REPAIR ACTIVITY

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ABSTRACT

This paper considers a group of S identical aircraft, each of which is partitioned into K parts which fail exponentially. The only way in which a failed aircraft can be repaired is by cannibalizing its out-of-commission parts from other failed aircraft. The evolution of the number of good aircraft over time is governed by the transient behavior of an absorbing Markov chain. We can therefore study this behavior by matrix multiplication although the computational problem grows large for $K \geq 3$. Some numerical results and some approximations are also provided.

1. INTRODUCTION

This paper presents a model of a system subject to cannibalization. The system is a simple special case of the very general coherent structure considered by Hirsch, Meisner, and Boll (HMB) in [3]. It differs, however, from the model of HMB in its treatment of time.

In Sec. 2 the model is described in terms of a specific example. The reader can no doubt imagine a number of other situations where the same model would be appropriate. We also make clear how our model treats time differently from the model of HMB. Section 3 presents the Markov chain transition matrix which characterizes our problem, while Sec. 4 explores the computational chore associated with using the model. In Sec. 5 and 6, we exhibit several modest numerical examples and suggest means of lessening the computational chore for larger problems. Section 7 contains a special technique which is feasible for small-scale problems.

Although this paper does not completely and explicity solve the problem it poses, we hope that the results given will provide quantitative insights into the effects of cannibalization which will lead to further work in the field.

2. THE PROBLEM

We consider a group of S identical aircraft operating under the following conditions:

(a) Every aircraft includes K parts or assemblies. The time-to-failure of the ith part is exponentially distributed with distribution function, $1 - e^{-\lambda_i t}$. Futhermore, the K parts are assumed to be noncritical items so that no matter how many of the K assemblies fail during a mission it is always possible for an aircraft to return safely to base.

(b) A failed part can neither be repaired nor resupplied. However, there is sufficient repair capability to cannibalize unfailed parts from other failed aircraft in an attempt to minimize the number of out-of-commission aircraft. We assume that initial supplies of spare parts are zero, although this condition can be relaxed as we indicate later.

(c) The aircraft fly missions of fixed length T in which every "good" aircraft flies every mission. We assume that there is enough time between consecutive mission takeoff times for all aircraft to

return from the previous mission and for all cannibalizations to be performed. We indicate below how the "every good aircraft flies every mission" rule can be modified.

The basic quantity which the analysis below will provide is the expected number of good aircraft after t missions, $t=1, 2, \ldots$, denoted by $E[A(t)]$. This information will, for example, allow us to measure the changes in force capability over time which cannibalization affords under the conditions above.

It is important to distinguish the way in which our model differs in its treatment of time from that of HMB. In our case, only those parts which happen to be in the air during a given period of time are subject to risk. Good parts lying in partially cannibalized aircraft on the ground are not subject to failure. In effect, each part carries its own clock which is turned on when the part is sent up into the air and turned off when the part is on the ground. In the paper of HMB, however, every part that has not previously failed is subject to risk. There is a single clock which is turned on at time 0, and while it runs all parts are subject to failure. In Sec. 6 below we use the above fact to identify one of the results of HMB as a lower bound for $E[A(t)]$.

3. THE MARKOV TRANSITION MATRIX

If we let $n_i(t)$ be the number of copies of part i which survive t missions, then $n(t) \equiv (n_1(t), \ldots, n_K(t))$ represents the number of parts of each type which survive t missions. This is the basic state vector of our process. As $n_i(t)$ is an integer and $0 \leq n_i(t) \leq S$ for $i=1, 2, \ldots, K$, the process has $(S+1)^K$ states for each t. However, for our purposes we can lump all states having one or more $n_i(t)=0$ into a single "zero" state, thereby reducing the number of states to S^K+1. It is readily seen that the process is Markovian and that its evolution from mission t to $t+1$ can be described by a one-step Markov chain transition matrix P of dimension S^K+1 by S^K+1. If the rows and columns are ordered lexicographically, P will be a lower triangular matrix with 0's above the principal diagonal.

It is convenient to label row and column m by d_m, where

$$d_1 = (\text{zero state});$$

$$d_2 = (1, \ldots, 1);$$

$$\vdots$$

$$d_{S^K+1} = (S, \ldots, S).$$

Then the matrix element p_{ij}, where $d_i=(i_1, \ldots, i_K)$, $d_j=(j_1 \ldots, j_K)$, and $M_i = \min_{l=1,\ldots,K} i_l$, is given by

$$\text{(2a)} \qquad \prod_{l=1}^{K} \binom{M_i}{j_l - i_l + M_i} (e^{-\lambda_l T})^{j_l - i_l + M_i} (1-e^{-\lambda_l T})^{i_l - j_l} \quad {}_l$$

if $i, j > 1$ and $j_l \leq i_l \leq j_l + M_i$, for all $l=1, \ldots, K$;

(2b) 0 if $i, j > 1$ but $i_l < j_l$ or $i_l > j_l + M_i$ for at least one $l=1, \ldots, K$;

(2c) 1 if $i=j=1$;

(2d) 0 if $i=1, j>1$;

(2e) $$1-\sum_{d=2}^{i} p_{id} \text{ if } i>1, j=1.$$

These entries become apparent if we realize that M_i copies of each part are tested on each mission and that each part is assumed to fail independently of other parts. Once the zero state is reached the "experiment" ends as all aircraft are out-of-commission.

4. THE COMPUTATIONAL CHORE

If we let $\Pi(t)$ be the row vector of unconditional state probabilities after t missions, then from Markov chain theory we know that

(3) $$\Pi(t)=\Pi(0)P^t, \qquad t=1, 2, \ldots,$$

where $\Pi(0)=(0, \ldots, 0, 1)$ as the process starts in state $(S, S, \ldots, S)$ and the states of $\Pi(\cdot)$ are ordered lexicographically as are those of P. Furthermore, P represents an absorbing Markov chain [4] with d_1 the absorbing state and all other states transient. From (3) we see that to go from $\Pi(t)$ to $\Pi(t+1)$ we must multiply $\Pi(t)$, a 1 by S^K+1 matrix, by P, an S^K+1 by S^K+1 matrix. This operation requires roughly $(S^{2K}/2)$ multiplications and the same number of additions. The major problem in using the model is one of computational feasibility as $(S^{2K}/2)$ can be a large number. For example, if $S=24$ and $K=3$, $(S^{2K}/2) \doteq 1.5\times 10^8$. Thus it appears that either S or K must be fairly severely restricted if we are to perform the computation in an acceptable amount of time; the following sections describe some numerical results which suggest ways of lessening the computational chore.

The first two moments of the number of good aircraft after t missions are given by

(4) $$E[A(t)]=\sum_{j=1}^{S^K+1} \Pi_j(t)M_j$$

and

(5) $$E[A(t)^2]=\sum_{j=1}^{S^K+1} \Pi_j(t)M_j^2,$$

where

$$M_j=\min_{l=1, \ldots, K} \{j_l \mid d_j=(j_1, \ldots, j_K)\}.$$

Since M_j is independent of t for each j, the computation of $E[A(t)]$ should proceed rapidly once $\Pi(t)$ has been determined. As the sole absorbing state d_1 of the Markov process has $M_1=0$ it is clear from (4) that $\lim_{t\to\infty} E[A(t)]=0$; this is of course what one would expect given the no-repair situation being modeled. The computation of $\Pi(t)$ would generally be iterated until $E[A(t)]$ was fairly close to zero.

5. NUMERICAL RESULTS FOR K=2

In order to simplify the computational task, we first limit ourselves to the case of two cannibalizable parts per aircraft. There remain, then, two dimensions to the problem: S, the number of aircraft under test, and $q \equiv (e^{-\lambda_1 T}, e^{-\lambda_2 T})$, the vector whose components are the probabilities of nonfailure of each part on one mission of length T. Table 1 presents results for q fixed and S variable, while in Table 2 q is variable and S fixed. In both cases the quantity of interest is $E[A(t)]$, the expected number of good aircraft after t missions.

TABLE 1

t \ S	Any S, no cannibalization; $(0.72)^t$	2	5	10	13
1	0.720	0.734	0.762	0.777	0.783
2	0.518	0.557	0.600	0.622	0.628
3	0.373	0.431	0.480	0.500	0.504
4	0.268	0.336	0.384	0.402	0.405
5	0.193	0.263	0.308	0.323	0.326
6	0.139	0.206	0.247	0.260	0.261
7	0.100	0.161	0.198	0.209	0.210
8	0.072	0.125	0.159	0.168	0.169
9	0.054	0.097	0.128	0.135	0.136
10	0.037	0.075	0.102	0.108	0.109
11	0.027	0.058	0.082	0.087	0.087
12	0.019	0.044	0.066	0.070	0.070
13	0.014	0.034	0.053	0.056	0.056
14	0.010	0.026	0.042	0.045	0.045
15	0.007	0.019	0.033	0.036	0.036

(Entries are $E[A(t)]/S$; $q=(0.9, 0.8)$. For the no cannibalization case $E[A(t)]/S=(0.72)^t$ results from the fact that 0.72 is the probability that one aircraft survives one mission and that we have a sequence of t binomial trials.)

From Table 1 we see that for a fixed value of t, the ability to cannibalize produces an increase in $E[A(t)]/S$ which also increases with S. However, we note that as S increases the quantity $E[A(t)]/S$

TABLE 2

t \ q	0.9, 0.8	0.8, 0.8	0.7, 0.7	0.6, 0.6	0.8, 0.5	0.5, 0.5	0.9, 0.4	0.5, 0.4
1	0.777	0.731	0.620	0.514	0.494	0.412	0.399	0.353
2	0.804	0.768	0.661	0.558	0.506	0.456	0.400	0.391
3	0.803	0.773	0.667	0.563	0.504	0.456	0.401	0.389
4	0.806	0.775	0.668	0.558	0.503	0.450	0.400	0.382
5	0.805	0.777	0.668	0.556	0.503	.....	.....	.....
6	0.802	0.776	0.667	0.550	.....	.....	.....	.....
8	0.803	0.776	0.660	.....	.....	.....	.....	.....
10	0.804	0.775	.....	.....	.....	.....	.....	.....
12	0.804	0.773	.....	.....	.....	.....	.....	.....
14	0.804	.....	.....	.....	.....	.....	.....	.....

(Entries are $E[A(t)]/E[A(t-1)]$; $S=10$.)

converges for values of S around 10. The entries for $S=13$ are not much different from those for $S=10$. Hence it seems that we can learn as much about $E[A(t)]/S$ by studying 13 aircraft as by studying 24 or 48 and the computational job is markedly easier. Of course, this conclusion is strictly valid only for $K=2$ and $q=(0.9, 0.8)$. A question of interest (as yet unanswered) is, "At which value of S does convergence occur, given a particular K and q?"

In Table 2 we exhibit the behavior of successive values of $E[A(t)]$ for a group of 10 aircraft and a variety of q-vectors. One fact is immediately apparent: For $t \geq 2$, $E[A(t)]/E[A(t-1)] \doteq \rho$, ρ a constant.

This is equivalent to saying $E[A(t)]/S \doteq \rho^{t-1}$, $t \geq 2$, which implies

$$E[A(t)]/S \doteq \rho_0(\rho)^{t-1}, \quad t \geq 1, \tag{6}$$

where

$$\rho_0 \equiv E[A(1)]/S.$$

This provides an immediate analogy to the no-cannibalization case, where we know that $E[A(t)]/S = \lambda^t$, $t \geq 1$, with $\lambda = q_1 q_2$ where $q = (q_1, q_2)$. (It was found that for $S = 5$ the geometric behavior was less stable than that characterized by Table 2 for $S = 10$).

The next section will suggest several possible ways around the computational problem.

6. APPROXIMATIONS FOR K > 2

In Sec. 5 we indicated that for $K = 2$, any $S \geq 10$ and any q we could approximate the behavior of $E[A(t)]/S$ by studying that of the case $S = 10$. A typical computer run on the 360/65 for $S = 10$ and $q = (0.9, 0.8)$ took approximately 7 sec. If K were increased to 3, the information in Sec. 4 would suggest a run time of roughly 10 min., still a feasible amount of time. Hence for $K = 2,3$ we can study a group of 10 aircraft and be fairly confident that the $E[A(t)]/S$ so obtained closely approximates that for any larger value of S and the same q.

For $K \geq 4$, however, studying even 10 aircraft becomes prohibitively costly as each time K is increased by 1, computation times are increased by a factor of 100. Thus, although the geometric behavior exhibited by $E[A(t)]/S$ in (6) would probably still hold, the problem of determining ρ_0 and ρ could no longer be handled by numerical means.

An approximation for $E[A(t)]$ is given by (7), with $q = (q_1, \ldots, q_K)$ and $\alpha = \min_i q_i$,

$$\frac{E[A(t)]}{S} \doteq \alpha^t, \quad t \geq 0. \tag{7}$$

Relation (7) is based on the imprecise reasoning that $A(t)$ is equal to the number of survivors of the part-type for which q_i is minimized.

The numbers in Table 2 suggest that the RHS of (7) is often an upper bound for $E[A(t)]/S$ rather than an approximation; however, column 5 of that table shows that this is not always so. For $t = 1$, the relationship $E[A(1)]/S \leq \alpha$ follows readily from $\min_{i=1,\ldots,K} X_i \leq X_i, \quad i = 1, \ldots, K,$

A lower bound for $E[A(t)]/S$ is given by (8), with $\beta = \prod_{i=1}^{K} q_i$,

$$\frac{E[A(t)]}{S} \geq \beta^t, \quad t \geq 0. \tag{8}$$

The referee suggested the following better lower bound, which can be derived from formulae 4.3 and 4.9 of [3]. This result can also be derived directly without reference to [3] and is based on the supposition that every surviving part be tested on every mission.

$$\frac{E[A(t)]}{S} \geq \frac{1}{S} \sum_{k=1}^{S} \prod_{i=1}^{K} \sum_{j=k}^{S} \binom{S}{j} q_i^{tj} (1 - q_i^t)^{S-j}. \tag{9}$$

The referee provided the results shown in Table 3 whose entries are the lower bound, (9), for the case shown in Table 1. Expression (9) appears to be an excellent lower bound in this instance.

TABLE 3

t \ S	2	5	10	13
1	0.734	0.759	0.777	0.782
2	0.554	0.598	0.620	0.626
3	0.423	0.473	0.495	0.501
4	0.323	0.374	0.396	0.401
5	0.247	0.296	0.316	0.320
6	0.187	0.233	0.252	0.256
7	0.142	0.184	0.201	0.204
8	0.106	0.144	0.160	0.163
9	0.080	0.113	0.127	0.130
10	0.059	0.088	0.101	0.103
11	0.044	0.068	0.080	0.082
12	0.032	0.052	0.063	0.065
13	0.024	0.040	0.049	0.052
14	0.017	0.030	0.039	0.041
15	0.013	0.023	0.030	0.032

7. A RECURSIVE RELATION FOR THE GENERATING FUNCTION OF THE STATE PROBABILITIES

The fact that the P-matrix, as given by expression (2), is lower triangular allows us to develop a recursive relation which, at least in theory, produces a closed form expression for the generating function of the state probability vector, $\Pi(t)$. Inverting these generating functions presents a barrier if S and K are large; however, we present the method as it might sometimes be desirable to study a problem with small S and K.

Setting $N=S^K+1$, we let $\{p_{ij},\ i, j=1, \ldots, N\}$ be the elements of the P-matrix. If $\Pi(t)=(\Pi_1(t), \ldots, \Pi_N(t))$ is the vector of state probabilities, we define the generating functions

$$\tilde{\Pi}_i(Z)=\sum_{t=0}^{\infty}\Pi_i(t)Z^t, \qquad i=1, 2, \ldots, N. \tag{10}$$

The basic Markov chain relation is

$$\Pi(t)=\Pi(t-1)P. \tag{11}$$

Using (10), (11), and the fact that $p_{ij}=0$, $j>i$, we can derive

$$\tilde{\Pi}_N(Z)=p_{NN}Z\tilde{\Pi}_N(Z)+1$$

$$\tilde{\Pi}_i(Z)=\sum_{j=i}^{N}p_{ji}Z\tilde{\Pi}_j(Z), \qquad i=N-1, N-2, \ldots, 1.$$

This is equivalent to

$$\tilde{\Pi}_N(Z)=1/(1-p_{NN}Z) \tag{12a}$$

$$\tilde{\Pi}_i(Z)=\left(Z\sum_{j=i+1}^{N}p_{ji}\tilde{\Pi}_j(Z)\right)/(1-p_{ji}Z), \qquad i=N-1, N-2, \ldots, 1. \tag{12b}$$

From (12a) we know at once that

$$\Pi_N(t) = (p_{NN})^t,\ t \geq 0.$$

Then (12b) can be solved recursively as $\tilde{\Pi}_i(Z)$ depends only upon $\tilde{\Pi}_j(Z)$, $j > i$. The difficulty in a large problem is that inverting $\tilde{\Pi}_i(Z)$ is not trivial as it was in (12a) because the generating function will have a large number of multiple roots. There do exist both exact and approximate methods for such problems, some of which are found in [2, Chap. XI]. We present a short numerical example to illustrate the technique:

$$S = 2$$

$$K = 2$$

$$q = (0.5,\ 0.4)$$

From expression (2), we find that

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.8 & 0.2 & 0 & 0 & 0 \\ 0.5 & 0.3 & 0.2 & 0 & 0 \\ 0.6 & 0.2 & 0 & 0.2 & 0 \\ 0.52 & 0.24 & 0.08 & 0.12 & 0.04 \end{bmatrix}$$

Then expression (12) produces

$$\tilde{\Pi}_5(Z) = 1/(1-0.04Z)$$

$$\tilde{\Pi}_4(Z) = (0.12Z)/(1-0.2Z)(1-0.04Z)$$

$$\tilde{\Pi}_3(Z) = (0.8Z)/(1-0.2Z)(1-0.04Z)$$

$$\tilde{\Pi}_2(Z) = (0.24Z)/(1-0.2Z)^2(1-0.04Z)$$

$$\tilde{\Pi}_1(Z) = (0.0016Z)(5+Z)(65-Z)/(1-Z)(1-0.2Z)^2(1-0.04Z).$$

If we use methods in [1] to invert these generating functions and then use the relation

$$E[A(t)] = 0\{\Pi_1(t)\} + 1\{\Pi_2(t) + \Pi_3(t) + \Pi_4(t)\} + 2\{\Pi_5(t)\}.$$

we find

$$E[A(t)] = \frac{3}{2}t(.2)^t + \frac{7}{8}(.2)^t + \frac{9}{8}(.04)^t,\ t \geq 0.$$

Thus we have found a closed form expression for $E[A(t)]$.

8. VARIATIONS OF THE MODEL

Several of the assumptions made in Sec. 2 can be relaxed without materially affecting the development outlined above. The only change is in the Markov transition matrix P:

(a) There may be an initial supply, s_i, of spares for part i, $i = 1, 2, \ldots, K$. In this case the possible states of part i are $0, 1, \ldots, s_i + S$ and the P-matrix has

$$\prod_{i=1}^{K} (s_i + S) + 1$$

rows and columns. This would make the computational task mentioned in Sec. 4 more difficult;

(b) The assumption that "every good aircraft flies every mission" can be replaced by the requirement that only Q aircraft, if available, fly every mission. This would be accomplished by substituting Q for M_i in expression (2a) whenever $Q < M_i$. If this were done, the P-matrix would be made more sparse while $E[A(t)]$ would approach zero less rapidly because the Markov process would take more time to work its way from the initial state to the absorbing state.

9. ACKNOWLEDGMENTS

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SOME ESTIMATES OF RELIABILITY USING INTERFERENCE THEORY

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ABSTRACT

According to "interference theory" of reliability, a component fails if the maximum stress exceeds the component's strength. Assuming that both these quantities are random and their distributions are normal, we obtain in this paper some point and interval estimates of reliability when the stress distribution is known and a few observations exist on component strengths.

1. INTRODUCTION

In recent years, many writers have attempted to analyze the reliability of a component by the application of probability arguments to a certain physical model of failure. According to this model (which has come to be recently known in the literature as "interference theory"), a component fails if at any moment the applied stress (or load) is greater than its strength or resistance. The stress is a function of the environment to which the component is subjected, and its value at any point of time is considered to be a random variable. The strength of the component is measured by the stress required to cause failure. Strength depends on material properties, manufacturing procedures, and so on. If the components under question are mass produced and their selection in a given system is assumed to be made at random, then the strength should also be considered a random variable. The reliability of a component during a given period $[0, T]$ is taken to be the probability that its strength exceeds the maximum stress during the entire interval. The above formulation of the reliability problem was first proposed by Birnbaum [1], who also gave a method for estimating the related probability. In recent years, the model has found an increasing number of applications in many different areas, especially in the structural and aircraft industries.

The stress to which a component is subject in its operating environment can often be predicted using the available technological knowledge about the mechanical, thermal and other relevant conditions of the system, and the way they interact with each other. In many cases mathematical models are available which allow one to compute deterministically the values of "stress" at different points of time corresponding to a given set of input values describing the initial conditions of the system. For example, as Church and Harris [2] have pointed out, in a missile flight these initial values may correspond to the propulsive force, angles of elevation, changes in atmospheric conditions, etc. Since these input values have statistical distributions associated with them, the calculated "stress" values will also be randomly distributed. Moreover, since it will be possible to generate a practically unlimited number of such stress values using simulation methods by varying the input values in accordance with their respective distributions and hence to reconstruct the stress distribution as accurately as desired, we might assume that the distribution of the maximum stress is known. The strength distribution cannot, however, usually be computed from such *a priori* considerations. It can only be estimated using

statistical methods from the results of tests carried out for this purpose. The number of such tests is likely to be small in most cases due to practical limitations.

Under the assumption that the distributions for strength and the maximum stress are both normal, we derive in this paper some point and interval estimates of reliability using "interference theory," when as above the stress distribution is known and the strength distribution is unknown, but a few observations on the latter are available. The point estimates are given in section 2, and the interval estimates are derived in section 3.

Owen, Craswell, and Hanson [7] considered a similar problem and gave confidence intervals for the reliability when both stress and strength are (a) jointly bivariate normally distributed and observations are in pairs, or (b) when they are independently normally distributed, but they have a common (but unknown) variance. Most recently, Church and Harris [2] considered the same problem and provided a procedure for obtaining confidence intervals, which they compare with that of Govindarajalu [3]. Their procedure, although empirically demonstrated to be superior to that of [3], is, however, inexact since it uses the asymptotic normal approximation of a given statistic and requires the substitution of the population mean and standard deviations by their observed sample values. Our procedure for obtaining the confidence interval is based on that given in [7] and, though exact in a certain fashion, is admittedly inefficient. Some comparisons of this procedure with that of Church and Harris [2] are given in section 3. Sampling experiments show that the average length of the interval associated with the Church and Harris procedure is smaller than that of our procedure although it appears that, in general, it fails to cover the true parameter value more often.

2. POINT ESTIMATES

Let X denote the maximum stress to which the component is subject over a given period of time. We assume that X is normally distributed with mean μ_1 and variance σ_1^2, and that both these values are known. Let Y denote its strength. We assume that Y is also normally distributed with unknown mean μ_2 and variance σ_2^2. In what follows, we shall first assume that σ_2^2 is known and then that it is unknown. No doubt the latter is the more realistic case. In terms of our definitions we readily see the reliability to be

$$p = Pr\{Y - X > 0\} = \Phi\left(\frac{\mu_2 - \mu_1}{\sqrt{\sigma_1^2 + \sigma_2^2}}\right), \tag{1}$$

where

$$\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-x^2/2} dx.$$

Let $Y_1, Y_2, \ldots, Y_m$ denote m mutually independent observations on the strength of the components. These values are obtained from bench tests. On the basis of these observations, we obtain below the minimum variance unbiased (MVU) and the maximum likelihood estimates of p.

2.1 Case 1: σ_2^2 known

Let Z be a normal random variable with mean 0 and variance σ_1^2, and let

$$I = \begin{cases} 1 & \text{if} \quad Y_1 + Z > \mu_1 \\ 0 & \text{if} \quad Y_1 + Z \leq \mu_1 \end{cases}. \tag{2}$$

Then

$$E\{I\} = Pr\{Y_1 + Z > \mu_1\} \tag{3}$$
$$= 1 - \Phi\left(\frac{\mu_1 - \mu_2}{\sqrt{\sigma_1^2 + \sigma_2^2}}\right)$$
$$= p.$$

It therefore follows from Blackwell-Rao and Lehmann-Scheffe theorems (See Chap. 3, [5]) that the conditional expectation $E\{I \mid \bar{Y}\}$ is the MVU-estimate of p, where $\bar{Y} = \sum_{i=1}^{m} Y_i/m$ is the sufficient statistic. Since moreover the conditional distribution of $Y_1 + Z$ given $\bar{Y}$ is normal with

$$E\{Y_1 + Z \mid \bar{Y}\} = \bar{Y}, \tag{4}$$
$$\mathrm{Var}\{Y_1 + Z \mid \bar{Y}\} = \sigma_1^2 + \sigma_2^2\left(1 - \frac{1}{m}\right),$$

the MVU-estimate of p in this case is given by

$$\hat{p}_I = E\{I_1 \mid \bar{Y}\} \tag{5}$$
$$= Pr\{Y_1 + Z > \mu_1 \mid \bar{Y}\}$$
$$= \Phi\left(\frac{\bar{Y} - \mu_1}{\sqrt{\sigma_1^2 + \sigma_2^2(1 - 1/m)}}\right).$$

The maximum likelihood estimate of p, on the other hand, is given by

$$\hat{p}_I = \Phi\left(\frac{\bar{Y} - \mu_1}{\sqrt{\sigma_1^2 + \sigma_2^2}}\right). \tag{6}$$

It is seen that as m becomes large, the two estimates become equivalent.

2.2 Case 2: σ_2^2 unknown

In this case the sufficient statistics is the pair $(\bar{Y}, S_2^2)$ where

$$S_2^2 = \sum_{i=1}^{m} (Y_i - \bar{Y})^2.$$

Therefore, using arguments identical to those of Case 1, it follows that in this case, $E\{I/\bar{Y}, S_2^2\}$ is the MVU-estimate of p. From [6] we find the conditional density of Y_1 given $\bar{Y}$ and S_2^2 to be

$$f(y|\bar{Y}, S_2^2) = \begin{cases} \sqrt{\dfrac{m}{m-1}}\,\dfrac{\Gamma\left(\dfrac{m-1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\dfrac{m-2}{2}\right)}\,\dfrac{1}{S_2}\left\{1 - \left(\dfrac{\bar{Y} - y}{S_2}\right)^2 \dfrac{m}{m-1}\right\}^{\frac{(m-4)}{2}}, \\ \qquad \bar{Y} - \sqrt{\dfrac{m-1}{m}}\,S_2 \leq y \leq \bar{Y} + \sqrt{\dfrac{m-1}{m}}\,S_2, \\ 0 \qquad \text{otherwise.} \end{cases} \tag{7}$$

Therefore the MVU-estimate of p in this case is

(8)
$$\begin{aligned}\hat{p}_{II} &= E\{I|\bar{Y}, S_2^2\} \\ &= \Pr\{Y_1+Z>\mu_1|\bar{Y}, S_2^2\} \\ &= \int_{\bar{Y}-\sqrt{\frac{m-1}{m}}S_2}^{\bar{Y}+\sqrt{\frac{m-1}{m}}S_2} f(y|\bar{Y}, S_2^2)\ \Phi\left(\frac{y-\mu_1}{\sigma_1}\right) dy.\end{aligned}$$

By use of the transformation $z=\frac{1}{2}+\frac{1}{2}\frac{\bar{Y}-y}{S_2}\sqrt{\frac{m}{m-1}}$, the integral on the right hand side of (8) reduces to

(9)
$$\hat{p}_{II}=\int_0^1 \frac{\Gamma(m-2)}{\Gamma\left(\frac{m-2}{2}\right)\Gamma\left(\frac{m-2}{2}\right)} z^{(m-4)/2}(1-z)^{(m-4)/2}\Phi(\alpha+\beta z)\,dz,$$

where
$$\alpha=\left(S_2\sqrt{\frac{m-1}{m}}+\bar{Y}-\mu_1\right)/\sigma_1,$$

and
$$\beta=-\frac{2S_2}{\sigma_1}\sqrt{\frac{m-1}{m}}.$$

It does not appear that this integral can be evaluated in a closed form; however, it can be evaluated very easily for particular examples by the use of numerical integration.

The maximum likelihood estimate for p in this case is

(10)
$$\hat{\hat{p}}_{II}=\Phi\left(\frac{\bar{Y}-\mu_1}{\sqrt{\sigma_1^2+s_2^2}}\right).$$

where
$$s_2^2=\frac{S_2^2}{m-1}.$$

$\hat{\hat{p}}_{II}$ is biased, but might be quite efficient when m is reasonably large. Also, it can be shown that for large values of m, the MVU-estimate $\hat{p}_{II}$ and the maximum likelihood estimate $\hat{\hat{p}}_{II}$ are equivalent.

3. INTERVAL ESTIMATES

As in the discussions in the preceding sections, we give the interval estimates of reliability for two separate cases (a) when σ_2^2 is known, and (b) when σ_2^2 is unknown. We derive here only the lower confidence limit because of its importance in the present application. An extension of our procedure to two-sided intervals will be obvious.

3.1 Case I: σ_2^2 known

Let K_α be such that $\Phi(K_\alpha)=\alpha$. Since $\bar{Y}$ is normal with mean μ_2 and variance σ_2^2/m, it follows that

(11)
$$\Pr\left\{\frac{\sqrt{m}(\bar{Y}-\mu_2)}{\sigma_2}\leqslant -K_\alpha\right\}=1-\alpha,$$

or
$$\Pr\left\{\frac{\mu_1-\mu_2}{\sqrt{\sigma_1^2+\sigma_2^2}}\leqslant\frac{\mu_1-\bar{Y}-K_\alpha\sigma_2/\sqrt{m}}{\sqrt{\sigma_1^2+\sigma_2^2}}\right\}=1-\alpha,$$

or
$$\Pr\left\{ p \geq \Phi\left(\frac{\bar{Y} - \mu_1 + K_\alpha \sigma_2 / \sqrt{m}}{\sqrt{\sigma_1^2 + \sigma_2^2}} \right) \right\} = 1 - \alpha.$$

Thus the 100 $(1-\alpha)$ percent lower confidence limit for p in Case I is

(12)
$$p_{I,L}(1-\alpha) = \Phi\left(\frac{\bar{Y} - \mu_1 + K_\alpha \sigma_2 / \sqrt{m}}{\sqrt{\sigma_1^2 + \sigma_2^2}} \right).$$

3.2 Case II: σ_2^2 unknown

The procedure suggested by Church and Harris [2] for obtaining the interval estimate in this case is based on the statistic

$$V = \frac{\bar{Y} - \mu_1}{\sqrt{\sigma_1^2 + s_2^2}}.$$

They have shown that V is asymptotically normal with mean $(\mu_2 - \mu_1)/\sqrt{\sigma_1^2 + \sigma_2^2}$ and variance

$$\sigma_V^2 = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}\left[\frac{1}{m} + \frac{(\mu_2 - \mu_1)^2 \sigma_2^2}{2(m-1)(\sigma_1^2 + \sigma_2^2)^2} \right].$$

Using normal approximation for the distribution of V and upon replacing the unknown values of μ_2 and σ_2^2 by their respective sample estimates $\bar{Y}$ and s_2^2, they obtain (the symbol $\doteq$ denotes approximately equal to)

(13)
$$\Pr\left\{ \frac{\mu_2 - \mu_1}{\sqrt{\sigma_1^2 + \sigma_2^2}} > V + K_\alpha \sigma_V \right\} \doteq 1 - \alpha,$$

or
$$\Pr\{ p > \Phi(V + K_\alpha \sigma_V) \} \doteq 1 - \alpha,$$

or
$$\Pr\{ p > \Phi(V + K_\alpha \hat{\sigma}_V) \} \doteq 1 - \alpha,$$

where

$$\hat{\sigma}_V^2 = \frac{s_2^2}{\sigma_1^2 + s_2^2}\left[\frac{1}{m} + \frac{(\bar{Y} - \mu_1)^2 s_2^2}{2(m-1)(\sigma_1^2 + s_2^2)^2} \right].$$

Since this procedure involves two approximations both of which may not be close in the non-asymptotic case, the above results may not be satisfactory in small samples.

We next suggest a procedure based on results given in section 4 of [7]. This procedure is "exact," but is admittedly inefficient because it does not make use of all the available information; however, the use of sampling adopted in this method seems to be the only way we can eliminate the nuisance parameter σ_2^2 without introducing any approximations, and we believe that the properties of the procedure might therefore be of interest. It seems possible that ad hoc procedures can be developed which use this sampling concept, but are more efficient than the procedure described here. Results obtained from simulation experiments given below seem to indicate that our procedure covers, in general, the true parameter value more often than the Church and Harris procedure, although the average interval obtained from the latter procedure is significantly shorter.

According to our procedure, we first draw, using random number tables, a sample of m independent observations form a normal distribution with mean 0 and variance σ_1^2. Having taken these m observations

we consider the transformed observations

$$W_i = Y_i + Z_i, \qquad i = 1, 2, \ldots, m.$$

It is clear that

$$E\{W_i\} = \mu_2 \tag{14}$$

and

$$\text{Var}\{W_i\} = \sigma_1^2 + \sigma_2^2.$$

Now let

$$\bar{W} = \frac{\sum_{i=1}^{m} W_i}{m} \tag{15}$$

and

$$s_W^2 = \frac{\sum_{i=1}^{m} (W_i - \bar{W})^2}{m-1}.$$

Then

$$\frac{\sqrt{m}(\mu_1 - \bar{W})}{s_W} = \left[\frac{\sqrt{m}\,(\mu_1 - \mu_2)}{\sqrt{\sigma_1^2 + \sigma_2^2}} + \frac{\sqrt{m}\,(\mu_2 - \bar{W})}{\sqrt{\sigma_1^2 + \sigma_2^2}} \right] \Big/ \frac{s_W}{\sqrt{\sigma_1^2 + \sigma_2^2}}, \tag{16}$$

is distributed as a non-central t with degrees of freedom $= m-1$ and the non-centrality parameter $\delta = \sqrt{m}(\mu_1 - \mu_2)/\sqrt{\sigma_1^2 + \sigma_2^2}$. Let the left hand side of (16) be denoted by T_{m-1}. It then follows from the formulas given in [4] that

$$\Pr\left\{ \frac{\sqrt{m}\,(\mu_1 - \mu_2)}{\sqrt{\sigma_1^2 + \sigma_2^2}} > \delta(m-1, T_{m-1}, \alpha) \right\} = 1 - \alpha, \tag{17}$$

where from Eq. (33) of [4], $\delta(m-1, T_{m-1}, \alpha)$ is defined as

$$\delta(m-1, T_{m-1}, \alpha) = T_{m-1} - \lambda(m-1, T_{m-1}, \alpha) \left[1 + \frac{T_{m-1}^2}{2(m-1)} \right]^{1/2} \tag{18}$$

and $\lambda(m-1, T_{m-1}, \alpha)$ can be obtained from Table IV of Johnson and Welch [4]. From (17) and (18) it now follows that

$$\Pr\left\{ p > \Phi\left(\frac{\delta(m-1, T_{m-1}, \alpha)}{\sqrt{m}} \right) \right\} = 1 - \alpha, \tag{19}$$

so that a lower $(1-\alpha)$ percent limit for p following this procedure is given by $\Phi(\delta(m-1, T_{m-1}, \alpha)/\sqrt{m})$.

We now exhibit the results obtained from 200 artificial sampling experiments comparing the above two procedures using the same set of random numbers. In Table 1 the Church and Harris procedure is denoted as Procedure A and the method suggested in this paper as Procedure B. The particular sample sizes shown in the table were chosen because of the comparative ease with which the values of $\delta(m-1, T_{m-1}, \alpha)$ can be evaluated from the existing tables.

TABLE 1. *Comparison of Confidence Interval Procedures*

$(\mu_1=0, \sigma_1=1; 1-\alpha=0.95)$

p	μ_2	σ_2	m	Number of one-sided confidence intervals out of 200 which covered true parameter value		Average length of 200 confidence intervals	
				Proc. A	Proc. B	Proc. A	Proc. B
0.90	1.43287	0.5	10	184	191	0.15	0.28
			37	183	195	0.12	0.19
	1.81246	1.0	10	171	187	0.19	0.26
			37	185	193	0.14	0.19
	4.05278	3.0	10	187	183	0.25	0.27
			37	187	191	0.17	0.18
0.95	1.83905	0.5	10	181	188	0.087	0.19
			37	188	190	0.067	0.10
	2.32624	1.0	10	184	194	0.12	0.21
			37	184	190	0.08	0.11
	5.20163	3.0	10	187	189	0.18	0.20
			37	185	189	0.10	0.11
0.99	2.60055	0.5	10	178	187	0.022	0.095
			37	186	193	0.015	0.038
	3.28946	1.0	10	182	194	0.045	0.097
			37	183	189	0.022	0.039
	7.35546	3.0	10	190	193	0.093	0.11
			37	192	188	0.034	0.039

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* * *

THE OPTIMAL BURN-IN TESTING OF REPAIRABLE EQUIPMENT*

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ABSTRACT

The subject of this paper is the utilization of the "infant mortality" or decreasing failure rate effect to improve the reliability of repairable devices. Decreasing failure rate implies the possibility that devices which exhibit it can be improved by "burn-in testing" of each unit. Such a test serves to accumulate operating time while shielded from the full costs and consequences of failure. A general formulation of the burn-in test decision for repairable devices is presented and some special cases are solved. A class of models, indexed by the degree of partial replacement present in the repair process, is considered and numerical results for the optimal policy are given for several members of that class. A comparison of those results reveals the profitability of testing increases with the complexity of the repairable device.

1. INTRODUCTION

The subject of this paper is the utilization of the "infant mortality" effect to improve the reliability of repairable devices. The term infant mortality denotes a decrease with age of the conditional probability of failure of a device and, consequently, an improvement in reliability with time as the device operates. This phenomenon is widely observed. For example, decreasing failure rate has been found to occur in electronic systems and components [12], biological systems [5], social systems such as business enterprises [20], and also, in mechanical systems. In the case of repairable equipments decreasing failure rate is often observed after the occurrence of failure and repair.

Decreasing failure rate implies the possibility that devices which exhibit it can be improved by "burn-in testing" of each unit. The term burn-in testing is taken to mean test stand operation under environmental conditions as close as possible to true operating conditions. The basic purpose of this test is to accumulate operating time while shielded from the full consequences and costs of failure. The measurement of reliability, while not a primary goal, may be a partial by-product.

The main theme of this paper is a general formulation of the burn-in test decision problem for repairable equipment. A continuous stage dynamic program is used to treat this sequential decision problem and some solutions are given for special cases. Space does not permit a full discussion of the class of models for which this decision structure applies; however, one broad family of models, indexed by the degree of partial replacement present in the repair process, is presented. Numerical results for the optimal test policy are given for several members of that family, and a comparison of those results reveals an increase in the profitability of testing as a function of the complexity of the repairable device.

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2. LITERATURE REVIEW

The distribution theoretic conditions under which the expected future life of a device increases with its age are shown by Watson and Wells [27]. Based upon the work of Knight [18], one stated result is that the expected future life of an aged device always exceeds that of a new one if the device has a monotonically decreasing failure rate.

Several authors have investigated the burn-in test problem for the case of unrepairable equipments. In that problem a failed device is considered worthless. The quantities investigated are the burn-in time required to achieve a specified expected future life of the surviving units. Watson and Wells [27] discuss these quantities for several failure distributions. A nonparametric method is considered in a paper by Lawrence [19]. Assuming only a monotonically decreasing failure rate, he derives bounds upon the burn-in time required to achieve a specified expected future life.

Recently, interest has developed in the burn-in testing of repairable devices. This potentially important problem requires some explicit assumptions about the nature of the relevant failure and repair processes. This necessity follows from the need to assess the relative worth of the repaired device. Kooharian and Young [28] consider a useful class of reliability models for repairable complex devices. The outstanding feature which they prove for this class is a special simplified form of the burn-in test decision policy. A similar model for the "debugging period" is given by Bazovsky [4].

Concern for the long run reliability improvement of complex systems is evidenced by a paper by Barlow and Scheuer [3] entitled "Reliability Growth During a Development and Testing Program." They attributed the improvement mainly to engineering changes.

In a report upon which this paper is based [7], the author presents some models of decreasing failure rate processes incorporating explicit consideration of the repair process. This union of failure and repair models appears to be a fruitful basis for the discussion of burn-in testing in view of the interaction between the two processes. A complete picture of the life of the device is given in terms of a stochastic process, usually with non-independent interfailure times.

3. ELEMENTS OF THE PROBLEM

The Unit Test

A unit test starts with the testing of a new unit and ends with the output of one tested unit. If a failure occurs during test, the test is interrupted, the unit is repaired, and the test then resumes. The time elapsed during repair is not counted.

The Failure Process for New Units

Given a population of indistinguishable new devices, a specified and constant environment in which they are intended to operate, and a set of conditions which define failure, a distribution function $F_0(t/\theta)$ with parameters θ and density $f_0(t/\theta)$ is utilized to represent the uncertainty which exists regarding t, the duration of operating time prior to the occurrence of failure for any one of the devices. The failure rate of that density, also known as the hazard rate, conditional probability of instantaneous failure, or force of mortality, is a very important function for determining the reliability characteristics of the device. For a distribution satisfying $F_0(0^-/\theta) = 0$, the failure rate, a function of time, is defined as

$$\mu_0(t/\theta) = \frac{f(t/\theta)}{[1 - F_0(t/\theta)]} \text{ for } t \geq 0.$$

It will be assumed throughout that $f_0(t/\theta)$ is a continuous function of t and that $F_0(0^-/\theta) = 0$.

Density of Future Time

The original density, f_0, represents the process only before testing begins. Suppose that at some time, τ, after the test starts no failure has been observed in $(0, \tau)$. Let t = future time to failure. Then the total time to failure is $(t+\tau)$. Of this total time to failure the only random part remaining is t. The relevant density function is that of future time to failure. This renormalized form of the density will be denoted here by

$$f(t/\theta, \tau) = \frac{f_0(t+\tau/\theta)}{[1-F_0(\tau/\theta)]}.$$

While this density function usually represents a larger family of densities than does $f_0(t/\theta)$, its failure rate is the same except displaced to the left by τ; $\mu_0(t+\tau/\theta)$.

The Repair Process

We consider that a failure during test will be followed by a repair of the failed unit. The rule which assigns a new density of time to next failure for the repaired unit will be called the "repair rule."

In many analysis it is assumed that the repaired unit has the same density of time to next failure as the new units. This practice ignores the information available from the failure and from the past history of the unit. Since it is possible to distinguish between new and repaired units we realize that it may be valuable to do so. This is particularly relevant for burn-in testing models since the purpose of testing is to create a difference between new and aged units.

The only restriction which we shall place upon the new density is that it be from the same family of density functions as is $f(t/\theta, \tau)$. Hence we allow τ to have the meaning of a parameter of the density rather than strictly the time since last failure.

The repair rule then is a transformation which determines which density of that family applies. The most general transformation to be considered here is one that allows the new parameter values to be a function of the parameter values which applied just prior to the failure. Hence the new parameter values are

$$\theta_R = \theta_R(\theta, \tau)$$

and

$$\tau_R = \tau_R(\theta, \tau),$$

where (θ, τ) are the parameter values just prior to failure.

These functions will be required to have the property that, when applied to any point in the space of allowed parameter values, they will produce a point which is also inside that space. Also, their continuity in τ will be required.

The repair rule results from an explicit model of the repair process. Examples of this are given in [7].

The Reliability Value Function

The reliability value function specifies the value that the tested unit would have if testing were terminated. It gives the value as a function of any possible test position (θ, τ). It will be denoted by: $U(\theta, \tau)$ = the values of a unit when testing is terminated and a time when the state of process is (θ, τ).

This class of functions, $U(\theta, \tau)$, includes all expectations of the future behavior of the unit, since it includes all the parameters of the density function of future life.

The Cost of Testing

The last remaining element of the problem is the cost of testing. In this we seek to represent the actual cost of using the test facilities. The cost of testing will be considered to consist of the following two components:

C_1 = The cost of repair of a unit which fails during test. When several independent failure modes are considered this definition can be expanded to include a different cost for each failure type.

C_2 = The cost per unit test time.

4. OPTIMIZATION BY DYNAMIC PROGRAMMING

The profit from a unit test is the terminal value (given by the reliability value function) minus the test costs incurred during the testing process. At an intermediate time during the test the optimal test policy does not depend upon the costs already incurred since they are fixed or "sunk" costs. The optimal policy is that which maximizes the expected profit of the remaining portion of the unit test.

The decision-making process is sequential; the decision to stop or continue is remade after each observation of the partial test results. Since partial test results are revealed continuously in time each time point is, in fact, a decision point. The basic repeated unit of the decision tree is shown in Figure 1.

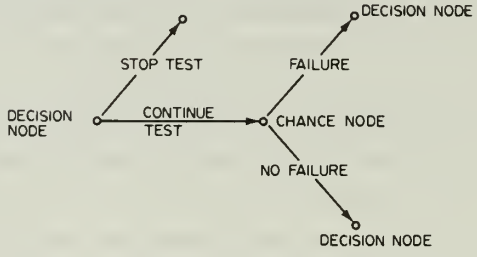

FIGURE 1. The basic unit of the Decision Tree

We define the optimal value function as

$V(\theta, \tau)$ = the expected value of the remaining portion of the unit test when the current state is (θ, τ) and when all future test decisions are optimal.

Let us consider a small positive time interval, h, starting at a time when the state of the process is (θ, τ). The optimal value function, $V(\theta, \tau)$, can be expressed in terms of the optimal decision at the start of the interval, the events which can occur during it, and the optimal value function at the possible resulting states. Let us consider the possibility of one event occurring in $(\tau, \tau+h)$ at time $\tau+\epsilon$, where $\epsilon < h$.

Since there are two possible decisions the optimal value function is equal to the maximum of two terms. The equation is

$$V(\theta, \tau) = \text{Max}\left\{ U(\theta, \tau), -C_2 h + \left[\frac{F_0(\tau+h/\theta) - F_0(\tau/\theta)}{1-F_0(\tau/\theta)}\right] \left[-C_1 + V(\theta_R(\theta, \tau+\epsilon), \tau_R(\theta, \tau+\epsilon))\right] + \left[\frac{1-F_0(\tau+h/\theta)}{1-F_0(\tau/\theta)}\right] V(\theta, \tau+h) \right\}.$$

The term $V(\theta, \tau)$ can be subtracted from both terms in the Max bracket. The resulting expression can be written as

$$0=\text{Max}\Big\{U(\theta,\tau)-V(\theta,\tau),\ h\Big\{-C_2+\frac{V(\theta,\tau+h)-V(\theta,\tau)}{h}$$

$$+\Big[\frac{F_0(\tau+h/\theta)-F_0(\tau/\theta)}{h(1-F_0(\tau/\theta))}\Big]\ [-C_1+V(\theta_R(\theta,\tau+\epsilon),\tau_R(\theta,\tau+\epsilon))-V(\theta,\tau+h)]\Big\}\Big\}.$$

Let us now observe that we have the form

$$0=\text{Max}\{U(\theta,\tau)-V(\theta,\tau),\ h\xi(h)\},$$

where $\xi(h)$ represents the interior of the second term. It should be noticed that the value of h has no effect through the term $\xi(h)$ since when $h\xi(h)<0$ the function $V(\theta,\tau)$ is determined by the equation $0=U(\theta,\tau)-V(\theta,\tau)$. In fact, for any positive h,

$$h\xi(h)<0 \iff \xi(h)<0,\ h\xi(h)=0 \iff \xi(h)=0.$$

Therefore, we can remove h and write the equation as

$$0=\text{Max}\{U(\theta,\tau)-V(\theta,\tau),\ \xi(h)\}.$$

The interval, h, can be considered as an inspection interval with the understanding that if a failure occurs anytime during the interval, its occurrence will not be detected until time, $\tau+h$, at which time the operation of the device will be resumed. If detection is not an important aspect of the problem, however, the limiting form of the equation is more useful. Provided only that the functions $f_0(\tau/\theta)$, $\theta_R(\theta,\tau)$, and $\tau_R(\theta,\tau)$ are continuous functions of τ everywhere in their range of definition, we can now let h approach 0. In the limit, we obtain the equation

$$0=\text{Max}\Big\{U(\theta,\tau)-V(\theta,\tau),\ \frac{\partial V(\theta,\tau)}{\partial\tau}-\mu_0(\tau/\theta)\,[V(\theta,\tau)-V(\theta_R,\tau_R)+C_1]-C_2\Big\}.$$

It is helpful to visualize the testing process as a point moving with time in the (θ,τ) space. As testing proceeds, τ (effective age of the unit) increases and θ remains constant at non-failure points in time. When a failure occurs, both θ and τ normally jump to a new value given by (θ_R,τ_R). The solution of the dynamic program divides the (θ,τ) space into two types of regions:

1. Continuation region where

   $(C1)$ $U(\theta,\tau)-V(\theta,\tau)\leq 0$, and

   $(C2)$ $\dfrac{\partial V(\theta,\tau)}{\partial\tau}-\mu_0(\tau/\theta)\,[V(\theta,\tau)-V(\theta_R,\tau_R)+C_1]-C_2=0.$

2. Termination region where

   $(T1)$ $U(\theta,\tau)-V(\theta,\tau)=0$, and

   $(T2)$ $\dfrac{\partial V(\theta,\tau)}{\partial\tau}-\mu_0(\tau/\theta)\,[V(\theta,\tau)-V(\theta_R,\tau_R)+C_1]-C_2\leq 0.$

These regions are separated by a boundary, $\tau^*(\theta)$, which represents the decision to terminate testing. A simple illustration of a process trajectory in state space is given in Figure 2.

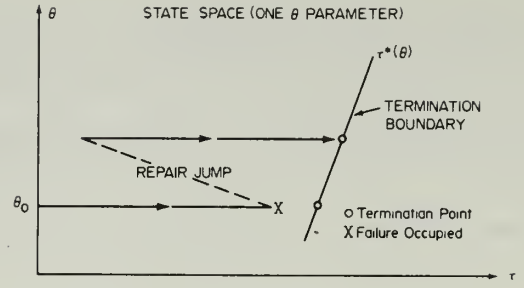

FIGURE 2. A process trajectory in state space

The following theorem helps to characterize the boundary in terms of the properties of $U(\theta, \tau)$.

THEOREM: For each θ, the optimal policy, $\tau^*(\theta)$, is restricted to the following possibilities:

a) $\tau^* = 0$ or ∞.

b) $\tau^* = \tau$ such that the derivative, $\dfrac{\partial U(\theta, \tau)}{\partial \tau}$, is discontinuous at τ.

c) $\tau^* = \tau$ such that $\dfrac{\partial U(\theta, \tau)}{\partial \tau}$ is continuous at τ,

$$\frac{\partial U(\theta, \tau)}{\partial \tau} = \frac{\partial V(\theta, \tau)}{\partial \tau}, \text{ and } \frac{\partial^2 U(\theta, \tau)}{\partial \tau^2} \leqslant \frac{\partial^2 V(\theta, \tau)}{\partial \tau^2} \text{ at } \tau.$$

PROOF: Assume that, for given θ, $V(\theta, \tau)$ and $\tau^*(\theta)$ are the optimal value function and decision boundary, and that

a) $0 < \tau^*(\theta) < \infty$

b) $U(\theta, \tau)$ has continuous first derivative, $\dfrac{\partial U(\theta, \tau)}{\partial \tau}$ at τ^*.

At a point $(\theta, \tau^* - \epsilon)$ (for $\epsilon > 0$) in the neighborhood to the left of τ^*, both $V(\theta, \tau)$ and $U(\theta, \tau)$ and their derivatives exist by assumption. Condition $C1$ gives

$$U(\theta, \tau^* - \epsilon) - V(\theta, \tau^* - \epsilon) \leqslant 0.$$

Expansion about the point $\tau = \tau^*$ gives

$$U(\theta, \tau^* - \epsilon) - V(\theta, \tau^* - \epsilon) = U(\theta, \tau^*) - V(\theta, \tau^*) - \epsilon \left[\frac{\partial U(\theta, \tau^* - \gamma\epsilon)}{\partial \tau} - \frac{\partial V(\theta, \tau^* - \gamma\epsilon)}{\partial \tau} \right]$$

for some $0 < \gamma < 1$. Conditions $C1$ and $T1$ then imply the following condition in terms of the left-hand derivatives

$$\frac{\partial V-(\theta, \tau^*)}{\partial \tau} \leqslant \frac{\partial U(\theta, \tau^*)}{\partial \tau}.$$

The left-hand derivative also satisfies Eq. $C2$

$$\frac{\partial V-(\theta, \tau^*)}{\partial \tau} - \mu_0(\tau^*/\theta)[V(\theta, \tau^*) - V(\theta_R, \tau_R) + C_1] - C_2 = 0.$$

In a neighborhood to the right of τ^* the conditions $T1$ and $T2$ give

$$\frac{\partial U+(\theta, \tau^*)}{\partial \tau} - \mu_0(\tau^*/\theta)[U(\theta, \tau) - V(\theta_R, \tau_R) + C_1] - C_2 \leqslant 0.$$

Subtracting the latter two conditions and using $T1$, we have

$$\frac{\partial V-(\theta, \tau)}{\partial \tau} \geqslant \frac{\partial U+(\theta, \tau)}{\partial \tau}.$$

Assumption C ensures that both one-sided derivatives of U are equal; hence we have the existence of V's derivative at τ^* and the equality

$$\frac{\partial V(\theta, \tau)}{\partial \tau}=\frac{\partial U(\theta, \tau)}{\partial \tau} \text{ at } \tau=\tau^*.$$

Furthermore, if the functions U and V have second derivatives at τ^* then a Taylor series expansion using both derivatives and the above equation yields

$$\frac{\partial^2 V(\theta, \tau)}{\partial \tau^2} \geqslant \frac{\partial^2 U(\theta, \tau)}{\partial \tau^2} \text{ at } \tau=\tau^*.$$

Hence part C of the theorem is proved. The possibilities excluded in assumptions a and b are parts a and b of the theorem.

The differential equation is a necessary condition upon the optimal value function. Its form and complexity will depend upon the nature of the θ space and repair functions. In the simplest cases the complete family of solutions of the differential equation can be obtained analytically. The parameters

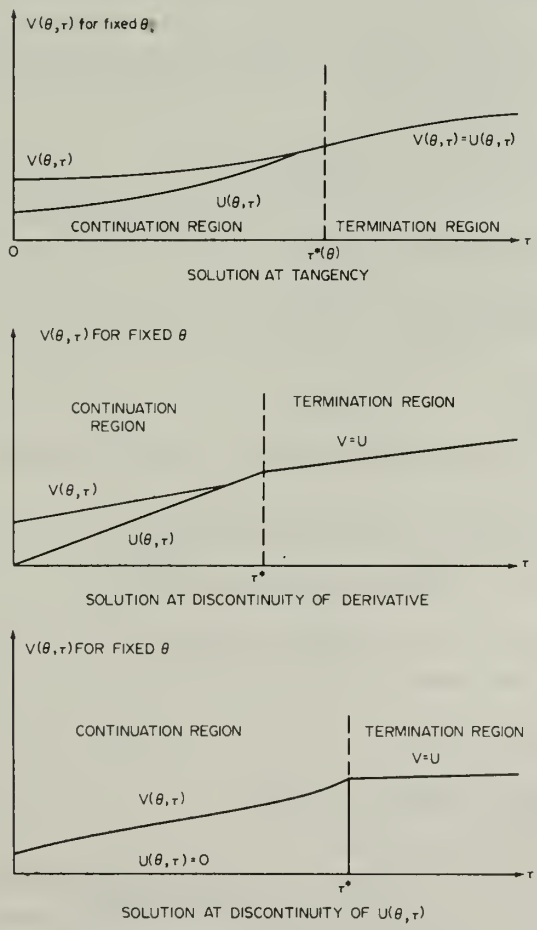

FIGURE 3. Possible types of solution

of its solution can then be evaluated using the above boundary conditions. This will be illustrated later for two special cases. Figure 3 illustrates the tangency condition and the two types of discontinuity conditions.

The method of optimization discussed here is a continuous stage dynamic program. Other mathematically similar problems in the literature, such as that of age replacement policy for stochastically failing devices which wear out, have been formulated as a dynamic program using a "renewal equation" form. For example, [11], [15], [16], and [17] use that form. Either form could be used here although this differential form is thought to have some computational advantage for this problem.

The Distribution of Remaining Test Time

When analyzing the feasibility of burn-in testing it is useful to know certain statistics of the unit test. Perhaps the most important of these are the moments of the test duration which will be considered here. Let t be the remaining test time when the test process is in state (θ, τ), $D(t/\theta, \tau)$ its density, and $D(S/\theta, \tau)$ be the Laplace transform of the density. An equation for $D(S/\theta, \tau)$, similar to the value function equation, can be derived using similar argument. By considering a small time interval, h, and the events possible within it, the following equation is derived:

$$D(S/\theta, \tau) = e^{-Sh} D(S/\theta_R, \tau_R)\, \mu_0(\tau/\theta)h + e^{-Sh} D(S/\theta, \tau)\, [1 - \mu_0(\tau/\theta)h] \text{ for } \tau < \tau^*(\theta).$$

By allowing h to approach zero, the limiting form obtained is

$$\frac{\partial D(S/\theta, \tau)}{\partial \tau} - \mu_0(\tau/\theta)\ [D(S/\theta, \tau) - D(S/\theta_R, \tau_R)\,] = SD(S/\theta, \tau) \text{ for } \tau < \tau^*(\theta).$$

The associated boundary condition is

$$D(S/\theta, \tau^*) = 1.$$

As a consequence, the kth moment of remaining test time, defined as

$$E_k(\theta, \tau) = \int_0^\infty t^k D(t/\theta, \tau)\, dt \text{ for } k = 0, 1, 2, \ldots,$$

obeys the following equation:

$$\frac{\partial E_k(\theta, \tau)}{\partial \tau} - \mu_0(\tau/\theta)\, [E_k(\theta, \tau) - E_k(\theta_R, \tau_R)\,] = -kE_{k-1}(\theta, \tau)$$

for $k = 1, 2, \ldots$, and t 1 t$^*(\theta)$. The boundary condition is

$$E_k(\theta, \tau^*) = 0 \text{ for } k = 1, 2, 3 \ldots .,.$$

The solution for these moments requires the knowledge of $\tau^*(\theta)$; then, since the $(k+1)^{st}$ equation involves the kth moment, the equations for the moments can be solved successively starting with $k = 1$.

5. EXTENSIONS OF THE FORMULATION

The Optimal Policy Under Discounting

In the analysis of processes which take place over an appreciably long time, thus tying up capital equipment and possibly delaying cash flows, it may be useful to account explicitly for "value of time" by optimizing the present discounted value of the process. Under this method, the values of all future

value transactions are discounted from face value according to the length of time into the future at which they take place. Hence, if the amount, U, is to be received at time, T, into the future, its present discounted value is $Ue^{-\beta T}$, where β is a positive discount factor.

The optimal policy and value function under discounting will be different from that already discussed; however, only a slight modification of the optimization equation is required to treat this case. The equation now appears as

$$0 = \text{Max}\left\{U(\theta, \tau) - V(\theta, \tau), \frac{\partial V(\theta, \tau)}{\partial \tau} - \beta V(\theta, \tau) - u_0(\tau/\theta)\left[V(\theta, \tau) - V(\theta_R, \tau_R) - C_1\right] + C_2\right\}.$$

The one additional term is $\beta V(\theta, \tau)$. All that has been said about the non-discounted optimal solution applies to this case also.

The effect of present value discounting will usually be to shorten the test. This can be expected since the terminal value is always received at the end of the test, whereas the test costs are spread out over the duration of the test. Therefore the expected present discounted value of a given policy is necessarily less than the expected undiscounted value of the same policy.

Burn-In Testing Under Uncertainty of the Failure Process

We have so far considered a known failure process. However, in many applications there exists a high degree of uncertainty about the underlying failure process. One method of relaxation of this assumption of complete knowledge of the density is to consider a known form of density function but with some unknown parameters. The state of knowledge about the values of such parameters is expressed, at any given time, by a joint density function over all allowable sets of their values. This prior density is modified by the observation of results of the testing process according to Bayes' Rule. The concept of a conjugate prior density, as described by Raiffa and Schlaiffer [25], is used here to enable a mathematically tractable formulation of this problem.

We suppose that the parameterization of the density includes parameter(s) ϕ which are unknown, but do not change with failure. They are now included explicitly in the density, $f(t/\theta, \tau, \phi)$. The "prior" density, that which describes the state of knowledge at a given time, is denoted $g(\phi/\psi)$. It is normalized over the range of possible ϕ values for any values of its parameters, ψ.

The density of life, unconditional upon the unknown ϕ, is

$$h(t/\theta, \tau, \psi) = \int_{\text{all } \phi} f(t/\theta, \tau, \phi) g(\phi/\psi)\, d\phi.$$

The prior density is chosen to be conjugate to $f(t/\phi, \tau, \phi)$ so that, after observation of a failure at time, t, following start-up of a unit of effective age, τ, the posterior density, according to Bayes' Rule,

$$\frac{f(t/\theta, \tau, \phi) g(\phi/\psi)}{h(t/\theta, \tau, \psi)}$$

has the same functional form in ϕ as does $g(\phi, \psi)$, but with parameters changed to $\psi_F = \psi_F(\theta, \tau, \psi, t)$. Hence the value of the parameters of the prior density is sufficient to describe the state of knowledge at any time.

The failure rate function which appears in the equation is

$$v(t/\theta, \tau, \psi) = \frac{h(t/\theta, \tau, \psi)}{\int_t^\infty h(x/\theta, \tau, \psi)\, dx}.$$

The state of the process under uncertainty is given by (θ, τ, ψ, x), where Ψ and τ are the values current at the time of last failure or start-up and x represents time since that event. Let this description be abbreviated by the definition

$$\theta' = \{\theta, \tau, \Psi\}.$$

The state after failure in state (θ', x) is

$$\theta'_R(\theta', x) = \{\theta, \tau + x), \tau_R(\theta, \tau + x), \Psi_F(\theta, \tau, \Psi, x)\}$$

$$x_R(\theta', x) = 0.$$

It is now apparent that the adaptive burn-in test problem, for one unit test, has the same form of optimization equation as previously described:

$$0 = \text{Max}\left\{U(\theta', x) - V(\theta', x), \frac{\partial V(\theta', x)}{\partial x} - v(x/\theta')[V(\theta', x) - V(\theta'_R, 0) - C_1] + C_2\right\}.$$

It will, however, be more difficult to solve because of the larger state space.

6. SOME SPECIAL CASES

There are some important special cases of burn-in test processes for which there are no θ type variables present; i.e., the failure density of a repaired unit is of the same form as that of some aged unit which never failed. In such cases, the effective age of the unit (τ) completely describes the satte of the process and the differential equation holds on a line. We now consider three such cases which encompass a range of possible repair rule cases.

Complete Replacement or Unrepairable Equipment ($\tau_R = 0$)

In the case of unrepairable equipment, a failure necessitates a complete replacement of the failed device by a new, untested one. Therefore the state of the process after failure is independent of the age attained by the old unit just prior to failure. This case is represented by the repair rule $\tau_R(\tau) = 0$.

The resulting linear first order differential equation,

$$\frac{dV(\tau)}{d\tau} - \mu_0(\tau/\theta)[V(\tau) - V(0)] = \mu_0(\tau/\theta)C_1 + C_2$$

is easily solved to give a family of all solutions parametric on one constant. For convenience, let

$$w(\tau) = \frac{C_1 F_0(\tau/\theta)}{[1 - F_0(\tau/\theta)]} + \frac{C_2 \int_0^\tau [1 - F_0(x/\theta)]\, dx}{[1 - F_0(\tau/\theta)]}.$$

Since the constant of solution is determined by the boundary conditions we write it as τ^*. The solution is then

$$V(\tau) = \begin{cases} V(\tau^*) - W(\tau^*) + W(\tau) & \text{for } 0 \leq \tau \leq \tau^* \\ U(\tau) & \text{for } t^* < \tau. \end{cases}$$

The value of τ^* is then determined as indicated by the previous theorem.

The independence of interfailure times allows easy solution for the Laplace transform of the density of remaining test time. It is

$$D(S/\tau) = \frac{Q(S, \tau)}{Q(S, \tau^*)}\, e^{-S(\tau^* - \tau)} \text{ for } \tau \leq \tau^*$$

where

$$Q(S, \tau) = \frac{1 - \int_0^{\tau} e^{-Sx} f_0(x/\theta)\, dx}{[1 - F_0(\tau/\theta)]}.$$

Although the solution for this case could easily be obtained by more direct arguments, it is presented here both to illustrate the method and to enable a comparison between unrepairable and repairable devices. We can already see that burn-in testing is somewhat inefficient for unrepairable devices since a failure results in the loss of all test effort expended up to the time of failure.

Superficial Failure ($\tau_R = \tau$)

In the previous case, the test time "invested" in the current unit was completely lost with a failure. It represents one extreme case the opposite of which is called here superficial failure and results in no loss of invested time. This case, represented by the repair rule $\tau_R(\tau) = \tau$, is important because of the simplification of the decision structure.

The differential equation for this case is

$$\frac{dV(\tau)}{d\tau} = \mu_0(\tau/\theta)\, C_1 + C_2.$$

The optimal value function in terms of τ^*, is

$$V(\tau) = \begin{cases} U(\tau^*) - C_1 \int_{\tau}^{\tau^*} \mu_0(x/\theta)\, dx - C_2(\tau^* - \tau) & \text{for } \tau \leq \tau^* \\ U(\tau) \text{ for } \tau \geq \tau^*. \end{cases}$$

The Laplace Transform for the density of remaining test time is easily found to be

$$D(S/\tau) = e^{-S(\tau^* - \tau)} \text{ for } \tau \leq \tau^*.$$

This is the transform of a unit impulse function located at $\tau = \tau$; consequently the length of remaining test time is not random. The sequential decision which remains unchanged in the face of partial test experience.

At first glance it seems counterintuitive that a failure leaves the device no better or worse than if no failure occurred; however, there are reasonable assumptions which lead to this repair rule. Kooharian and Young [28] present a class of such models. Similarly, Cozzolino [7] discusses a special case of that class, based upon the concept of production defects initially present in the device. Bazovsky [4] gives a similar debugging model. Because of its special deterministic nature, the decision policy is easy to implement. Also, as we shall see, this model results in a relatively profitable burn-in test.

Partial Renewal ($\tau_R = \alpha\tau$).

The repair rule $\tau_R(\tau) = \alpha\tau$ for $0 < \alpha < 1$ represents an intermediate degree of replacement. Consider, for example, a device composed of N indistinguishable and essential components. Failure results from failure of any one of the components. Repair is then accomplished by replacement of that failed component by a new, untested component. The test time previously accumulated on all other components is retained. This situation may be represented by a partial loss of effective age, where α in the repair rule is a fraction of the time retained; $\left(\frac{N-1}{N}\right)$ in this case. A more detailed discussion of this model is given in [7].

The differential equation for this class of repair rules is

$$\frac{dV(\tau)}{d\tau}-\mu_0(\tau/\theta)\,[V(\tau)-V(\alpha\tau)\,]=\mu_0(\tau/\theta)C_1+C_2.$$

The combination of differential and difference operators present makes analytic solution difficult for most failure rate functions of interest. However, approximate solutions based upon numerical integration are easily obtained. Since all solutions are within an additive constant of each other, the boundary conditions are easily satisfied.

A Specific Example

Some numerical results, based upon the above family of repair rules, were obtained in order to illustrate how the optimal test policy depends upon the repair rule parameter, α. The specific model used is based upon the density function,

$$f_0(t/a,\,b)=\frac{b}{a}\left(\frac{a}{a+t}\right)^{b+1}\qquad t\geqslant 0,\ b>0,\ a>0.$$

The failure rate associated with this density is monotonically decreasing,

$$\mu_0(t/a,\,b)=\frac{b}{a+t}.$$

The value of a tested unit is taken to be a function, $R(t)$, of its first life after test, t, as shown in Figure 4.

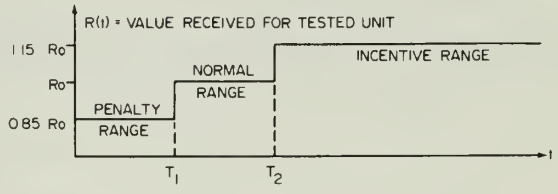

FIGURE 4. Value vs performance

The reliability value function is consequently the expected value of this function as of test termination and is a continuous function of age,

$$U(\tau)=\int_0^\infty R(t)f(t/\theta,\,\tau)\,dt$$

$$=R_0\left[0.85+0.15\left(\frac{a+\tau}{a+\tau+T_1}\right)^b+0.15\left(\frac{a+\tau}{a+\tau+T_2}\right)^b\right].$$

The parameter values chosen for the example are $a=72.0$, $b=1.1$, $C_1=1.0$, $C_2=0.01$, $R_0=90.0$, $T_1=500$, and $T_2=1000$. The mean life is found to be 720 units and the value of an untested unit is $U(0)=78.57$. The optimal test policy and value function were found for several values of repair rule parameter, α. The optimal policy, τ^*, is tabulated for these α values in Table 1, along with other quantities of interest. Figure 5 gives the optimal value function for one α value; $\alpha=7/8$. Figure 6 shows how the expected test profit, $V(0)-U(0)$ increases with a.

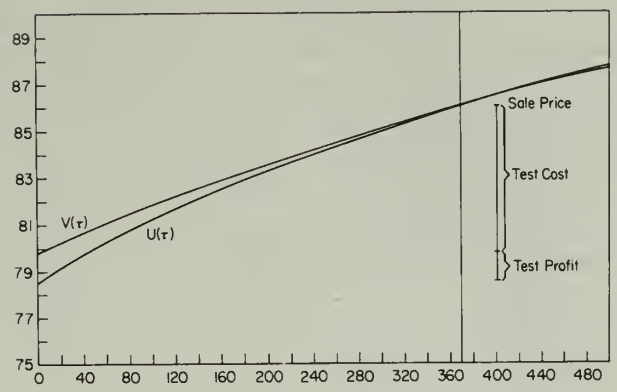

FIGURE 5. The optimal value function $V(\tau)$ for $\alpha = 7/8$

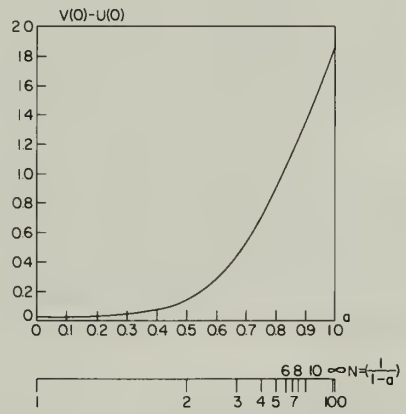

FIGURE 6. Test profit as a function of α; Case 3

The parameter, α, indexes a family of partial renewal models; the fraction of the device which is renewed with repair of a failure is $(1-\alpha)=1/N$. As a ranges from 0 to 1 the model ranges from complete replacement ($\alpha=0$) to the superficial failure case $\alpha=1$, and is therefore capable of representing a wide range of repair behavior. The computations show that as the degree of partial renewal decreases (α increases), the coefficient of variability of the total test duration decreases, as would be expected from previous results. As the test duration becomes more deterministic the optimal test policy, τ^*, lengthens. The expected profit due to testing increases and, in addition, the ratio of expected test profit to expected test cost increases, showing an inherently greater profitability as the renewed portion of the device decreases. This result conforms to our intuitive expectation based upon the previous results; as α increases a larger portion of the invested test time is retained at failure. Consequently, failure becomes less of a risk and the test process becomes closer to a deterministic one.

TABLE 1. *Optimal Test Policy for a Class of Repair Models*

α	$N=(1-\alpha)^{-1}$	Optimal policy τ^*	Expected value		Expected profit from test $[V(0)-U(0)]$	Ratio (%) of test profit to test cost $\left[\frac{V(0)-U(0)}{V(\tau^*)-V(0)}\right]$	Expected duration of unit test	Coefficient of variability of duration of unit test
			Before test $V(0)$	After test $V(\tau^*)$				
0	1	14.5	78.595	78.979	0.0249	6.5	15.7	0.28
0.5	2	80.0	78.70	80.68	0.13	6.6	97.5	0.26
0.6	2.5	140.0	78.83	82.04	0.26	8.1	174.2	0.25
0.7	3.3	220.0	79.09	83.63	0.52	11.5	268.4	0.20
0.75	4	261.0	79.27	84.36	0.70	13.8	310.9	0.17
0.8	5	304.2	79.47	85.07	0.90	16.1	351.7	0.14
0.875	8	370.0	79.82	86.05	1.25	20.1	406.4	0.09
0.9	10	390.0	79.94	86.32	1.37	21.5	420.6	0.07
1.0	∞	470.3	80.43	87.36	1.86	26.9	470.3	0.0

7. CONCLUSIONS

The analysis of the burn-in testing of repairable devices requires explicit information about the repair process in order to evaluate repaired devices. A class of repair models, having a form tractable with respect to the sequential decision problem for optimization of the burn-in test, is described by a repair rule of the form, $\theta_R(\theta, \tau)$, $\tau_R(\theta, \tau)$, specifying the parameters of a density function within the family of the renormalized form of the new unit density. This family of densities has the same functional form of failure rate.

The burn-in testing optimization, when treated as a continuous stage dynamic program, results in a differential equation similar to equations for the state probability of a stochastic process. In two important special cases, the differential equation is of the first order in total derivatives and consequently is easily solved and fit to the boundary conditions. In more complex cases it has a differential-difference nature.

The wide differences in the expected test profit under different repair models of the partial renewal family leads to the conclusion that the explicit modeling of the failure-repair process is highly important for the burn-in testing of repairable equipment. While this result is based upon a specific class of models, it appears likely that the greater profitability of burn-in testing of repairable over unrepairable devices may prevail over a much wider class of models.

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GOAL PROGRAMMING IN ECONOMETRICS

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1. INTRODUCTION

Optimization models have been suggested as an alternative to simultaneous equation models for use in dealing with the problems of policy formulation and decision making at the level of the national economy (Theil [11]; Holt [5]; van Eijk and Sandee [14]; and Hickman [4]). In simultaneous equation models a policy maker prescribes a set of fixed values for goal or target variables and these are interrelated with a set of instrument variables by means of a system of simultaneous linear equations. The policy maker then solves this system (in which the number of equations or restraints is usually required to be equal to the number of unknowns) for the values of the instrument variables. It has been emphasized (Tinbergen [12], [13]) that the policy maker's welfare function is not explicit in this model but is implicit in the selection of the goal and instrument variables. In the optimization approach one develops a welfare (preference) function which is optimized with respect to a set of econometric restraints. As is usual with such models, there can be many feasible solutions, and the optimization process then chooses a best or optimal solution from the set of feasible solutions.

In the Theil optimization model (Theil [11]) a preference function is developed in which both goal and instrument variables appear. A desirable configuration of goal and instrument variables, not necessarily satisfying the constraints, is specified by the policy maker, and a quadratic function in deviations between the desired and feasible values of the variables is optimized (minimized). Although the quadratic function allows the policy maker to take account of a decreasing marginal rate of substitution between any two variables (at least over an appropriate subset of the domain of the function), it has the disadvantage of assigning the same penalty to overattainment of a goal as to underattainment (Hickman [4], p. 6). A policy maker, for example, might not want to accept underattainment of a goal, but might be indifferent as to whether the goal was met or exceeded.†

In this paper, we introduce an optimization model which differs from that of Theil in several ways. It permits overattainment and underattainment of any goal, y_i^*, to be weighted either equally or differently in the decision maker's preference function. The model can be regarded as a special type of linear optimization model (as we make clear below) so it is easy to handle computationally and a wide range of parametric or sensitivity analyses can be implemented on a computer. These analyses can

†Still another approach has been developed (Spivey and Tamura [9]), that of using a generalized simultaneous econometric model (GSEM), where the word generalized is used to emphasize that the number of equations or constraints is not required to be equal to the number of variables. The GSEM can be regarded as a link between conventional simultaneous equation models and the Theil model since, like the former it consists of simultaneous linear equations and like the latter a GSEM can be shown to have a solution which is identical with an optimal solution to a Theil type optimization model.

display interactions between alternate optimal policies and goal attainment and can assist the policy maker in studying explicitly many of the trade offs between underattainment, attainment, and overattainment of selected goals. Such sensitivity analyses are not available for either the Theil model or Tinbergen type simultaneous equation models. The model can also easily accommodate explicit restrictions on instrument variables in the form of nonnegativity and upper and lower bounds.

The model is that of goal programming, originally developed by Charnes and Cooper in another context (see Charnes and Cooper [1]; also Charnes, Cooper, and Ijiri [2]; Ijiri [6]). We develop an application of this model using Klein's Model I of the United States economy as a basis (Klein [7]) and display in numerical form and interpret a sequence of decision making implications of several classes of the sensitivity analyses that can be performed with this model.

2. GOAL PROGRAMMING AND GOAL ATTAINMENT

A reduced form of a linear econometric model can be written

$$y = Rx + s, \tag{1}$$

where y is an n by 1 vector of goal or target variables, x is an m by 1 vector of instruments, R is an n by m (real) matrix of impact multipliers, and s is an n by 1 vector of constant or additive terms. If $m = n$, this is the reduced form of a conventional system of simultaneous linear equations; if $m \neq n$, then (1) is an example of a generalized simultaneous econometric model, in which $m = n$ is a special case (Spivey and Tamura [9]).

Let y^* be the goal vector prescribed by the policy maker and consider the model

$$y^* = Rx + s, \tag{2}$$

where R and s are assumed to be known. If R is singular, then (2) either has no solution or infinitely many solutions. If R is non-singular, then (2) always has a unique solution; however, it may not be acceptable to the policy maker because one or more components violate restrictions which are present in the mind of the policy maker, but are not explicitly a part of (2)—for example, nonnegativity.†

We reformulate (2) as

$$\begin{aligned} y^* - s &= Rx - Iz^+ + Iz^- \\ x, z^+, z^- &\geqq 0, \end{aligned} \tag{3}$$

where I denotes an n by n identity matrix, R is the n by m matrix introduced in (2), and z^+ and z^- are n by 1 (unknown) deviation or discrepancy vectors. The ith components z_i^+ of z^+ and z_i^- of z^- measure the deviation upwards and downwards, respectively, of feasible y_i values from the corresponding goal values y_i^*; in other words, the ith constraint of (3) can be written

$$y_i^* - s_i = R_i x - z_i^+ + z_i^- \qquad (i = 1, \ldots, n), \tag{4}$$

where R_i denotes the ith row vector of the matrix R and s_i the ith component of the vector s. Note that a nonnegativity restraint on x is explicit in this formulation.

If the policy maker can find a feasible instrument vector x for which both z_i^+ and z_i^- in (4) are zero, then the goal y_i^* can be attained exactly. On the other hand, if there is no feasible vector x for which $z_i^+ = z_i^- = 0$ in (4) he cannot attain the prespecified goal y_i^*. He can, however, find a feasible x that will

†If implicit restrictions on x are violated in the case R is nonsingular, Tinbergen [12], [13] would suggest model reformulation by selecting another goal vector $y^{*\prime}$ in an *ad hoc* manner, substituting $y^{*\prime}$ for y^* in (2), and again solving for x.

allow him to come as close as possible to y^* in a sense we will soon make clear. Now consider the following goal programming model

(5) $$\text{minimize } e^T z^+ + e^T z^-,$$

subject to

(6) $$y^* - s = Rx - Iz^+ + Iz^-$$
$$x, z^+, z^- \geqq 0,$$

where e^T is an n by 1 row vector† each of whose components is 1.

The z_i^+ and z_i^- $(i=1, \ldots, n)$ enter into the objective function (5) with a positive coefficient so the minimization process reduces the values of these variables in an optimal solution to the lowest feasible level and (5) attains a minimum value of zero if and only if both z^+ and z^- are the zero vectors (if and only if $z_i^+ = z_i^- = 0$ for each i). The equations (6) clearly have feasible solutions because the rank of the matrix I assures that (6) can be solved for the z's alone.

The advantage of reformulating (2) as a goal programming problem should now be clear: if (5) has a minimum value of zero, then (2) has a solution vector x which enables the policy maker to attain his prespecified goals y_i^* $(i=1, \ldots, n)$ exactly. If it does not, then (2) is not solvable for a feasible instrument vector x. In the latter case, however, an optimal solution to (5) and (6) will disclose (*a*) how close the policy maker can get to the desired goal vector y^*, and (*b*) whether this approximation is on the high side or low side for each individual goal, y_i^*.

The statement (*b*) can be made, and nonsense values of z_i^+ and z_i^- avoided, because the vectors in the system of restraints (6) associated with z_i^+ and z_i^- are negatives of each other. It is impossible for both z_i^+ and z_i^- to assume positive values in an optimal solution because the basis property of linear independence is preserved in the simplex solution procedure. Denote an optimal solution to (5) and (6) as

(7) $$\begin{pmatrix} x_0 \\ z_0^+ \\ z_0^- \end{pmatrix},$$

where $x_0^T = (x_{1,0}, \ldots, x_{m,0})$, $z_0^{+T} = (z_{1,0}^+, \ldots, z_{n,0}^+)$ and $z_0^{-T} = (z_{1,0}^-, \ldots, z_{n,0}^-)$. There are exactly three possibilities in an optimal solution to (5) and (6):

(i) $$z_{i,0}^+ = z_{i,0}^- = 0,$$

(ii) $$z_{i,0}^+ > 0, \; z_{i,0}^- = 0,$$

(iii) $$z_{i,0}^+ = 0, \; z_{i,0}^- > 0.$$

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† The superscript T denotes a transpose. All vectors will be regarded as column vectors unless this superscript is used. The model (5) and (6) can also be written

$$\text{Minimize } [0, e^T, e^T] \begin{pmatrix} x \\ z^+ \\ z^- \end{pmatrix}$$

Subject to

$$y^* - s = [R, -I, I] \begin{pmatrix} x \\ z^+ \\ z^- \end{pmatrix}$$

$$\begin{pmatrix} x \\ z^+ \\ z^- \end{pmatrix} \geqq 0.$$

Case (i) means that the ith goal y_i^* is attained exactly whereas cases (ii) and (iii) indicate that a deviation, upwards and downwards respectively, must be accepted by the policy maker desiring a nonnegative solution to (2).

The loose phrase, "how close the policy maker can get to the goal vector y^*," is given a precise meaning in goal programming because in (5) and (6) one is minimizing

$$\sum_i |y_i^* - (R_i x + s_i)|, \tag{8}$$

where the vertical bars denote absolute value. An optimal solution x_0 yields values of the discrepancy variables z_i^+ and z_i^- which have the following effects:

(i′) $z_{i,0}^+ = z_{i,0}^- = 0$ if $y_i^* = R_i x_0 + s_i$;

(ii′) $z_{i,0}^+ = R_i x_0 + s_i - y_i^*$, $z_{i,0}^- = 0$ if $y_i^* < R_i x_0 + s_i$;

(iii′) $z_{i,0}^- = y_i^* - (R_i x_0 + s_i)$, $z_{i,0}^+ = 0$ if $y_i^* > R_i x_0 + s_i$.

This can be stated in still another way. Given an x_0 in an optimal solution (7), the corresponding target configuration y_0, which occurs when x_0 is used in (2), is

$$y_0 = Rx_0 + s. \tag{9}$$

If case (i) occurs, $y_0 = y^*$ as desired, but in either case (ii) or (iii) $y_0 \neq y^*$. In the latter cases y_0 is the closest vector to y^* where closeness is measured by the absolute value metric or distance measure (this metric is sometimes called the L_1 metric). That is, y_0 is an optimal solution to the problem

$$\text{minimize} \sum_{i=1}^{n} |y_i^* - y_i|, \tag{10}$$

subject to

$$y = Rx + s,\ x \geqq 0. \tag{11}$$

The measure of closeness can be generalized by using an L_p metric and minimizing

$$\left(\sum_i |y_i^* - y_i|^p \right)^{1/p}, \tag{12}$$

subject to the restraints (11), where p is any integer greater than or equal to one. If $p = 2$, this is the same as minimizing the quadratic function $\Sigma (y_i^* - y_i)^2$ subject to the restraints (11), which is the form of the objective function in the Theil optimization model. The model in this case selects a feasible vector y which is closest to y^* in the sense of Euclidean distance or ordinary least squares.

The goal programming model (5) and (6) can be augmented so as to contain political and institutional as well as nonnegativity restrictions on the instrument variables $x_i (i = 1, \ldots, m)$. Such restrictions, called boundary conditions by Tinbergen ([12], [13]), must be dealt with implicitly in the use of his models, but can be incorporated as upper and lower bounds on x_i. These offer no difficulty in a linear programming model. For example, we could

$$\text{minimize } e^T z^+ + e^T z^- \tag{5}$$

subject to

$$y^* - s = Rx - Iz^+ + Iz^- \tag{6′}$$

$$x_i \geqq L_i \qquad i \epsilon S,$$
$$x_k \leqq U_k \qquad k \epsilon T,$$
$$x, z^+, z^- \geqq 0,$$

where S and T are any subsets of the set of integers $\{1, \ldots, m\}$ and L_i and U_k are specified lower and upper bounds, respectively. The model (5) and (6′) can be regarded as a decision model which explicitly takes account of all relevant restraints, both econometric and political or institutional.

3. WEIGHTING DISCREPANCY VARIABLES

In addition to determining a feasible vector y_o which is closest to the desired vector y^* in terms of the L_1 metric, a policy maker might regard attainment of certain goals as more important than that of others and might wish to use different weighting factors on deviation variables to reflect this. One can accordingly replace e^T in the objective function (3) by other nonnegative weighting vectors u^T and v^T. This gives the following objective function instead of (5),

$$u^T z^+ + v^T z^- \tag{13}$$

which is minimized subject to (6). Also,

$$u^T = v^T = e^T$$

is a special case in which each deviation variable is given a weight of one. Thus if a policy maker desires to attain a given goal y_i^* more than others he can attach larger weights u_i and v_i to the corresponding deviations z_i^+ and z_i^-, respectively. If he would accept an overattainment of y_j^*, but cannot tolerate any underattainment, he could set $u_j = 0$, but would make v_j large enough to prevent z_j^- from assuming a positive value in an optimal solution.† In this case, the model will allow the policy maker to impose the following condition

$$y_j^* \leqq R_j x + s_j, \tag{14}$$

i.e., we can accommodate an inequality restraint if desired—a feature which is not incorporated in the Theil optimization model.

Moreover, one can parameterize the objective function in a goal programming model to see how an initial optimal deviation package—say, one in which some goals are attained exactly and some are not attained initially—varies as different nonnegative weights are attached to the deviation variables. This can be contrasted with the effect of a weighting factor in a Theil model (L_2 metric model), which permits a delicate control of the *size* of the corresponding deviation symmetrically above or below the goal vector y^* (for an empirical study of the difference in effects of the use of L_1 metric and L_2 metric, see Tamura [10]).

4. SOME EXAMPLES OF DECISIONS IN GOAL PROGRAMMING MODELS

The classical Klein Model I (Klein [7]) will be used to present some numerical examples of goal programming decisions. Model I in reduced form is shown below; actual values have been given to the lagged variables and the trend variable so that model can be used in examining 1933 policy decisions.††

†For any nonnegative vector of coefficients $[u^T v^T]$, (13) has a lower bound of zero in a minimization problem. Only models with nonnegative coefficients in the objective function will be considered in this paper.

††This model was also used for illustrative purposes by Theil [11] and Spivey and Tamura [9].

Consider

$$\text{(15)}\qquad \begin{pmatrix} C \\ P \\ W_1 \\ I \\ Y \\ K \end{pmatrix} = \begin{pmatrix} 0.6659 & -0.1883 & 0.6705 \\ -0.2238 & -1.2808 & 1.1188 \\ -0.1620 & -0.2033 & 0.8102 \\ -0.0517 & -0.2959 & 0.2584 \\ 0.6142 & -1.4842 & 1.9289 \\ -0.0517 & -0.2959 & 0.2584 \end{pmatrix} \begin{pmatrix} W_2 \\ T \\ G \end{pmatrix} + \begin{pmatrix} 37.5812 \\ 9.0764 \\ 23.0109 \\ -5.4938 \\ 32.0873 \\ 201.6060 \end{pmatrix}$$

where C = consumption expenditures, P = corporate profits, W_1 = private wages, I = private investment, Y = national income, and K = end of year capital stock are endogeneous variables and W_2 = government wages, T = tax revenue, and G = government expenditures are exogeneous variables (all variables are measured in terms of billions of 1934 dollars). The constant terms are calculated under the assumption that the random disturbances are forecast accurately.

EXAMPLE 1. **Parameterization of Weights:** Suppose that the goal vector selected by a policy maker is $y^T = (C, P, W_1, I)$ and the instrument vector is $x^T = (W_2, T, G,)$. Then in the manner of Tinbergen and Theil we can formulate a linear econometric model in these vectors y and x by extracting the first four equations from the reduced form (15).

Let the goal values be set as a uniform 10 percent increase over the actual values of 1932 for each goal variable, i.e., $y^{*T} = (50.16, 7.7, 31.9, -5.58)$. Then we use the following goal programming model to examine 1933 policy decisions,

$$\text{(16)}\qquad \text{minimize } z_c^+ + z_c^- + z_p^+ + z_p^- + z_{w_1}^+ + z_{w_1}^- + z_I^+ + z_I^-$$

subject to

$$\text{(17)}\qquad \begin{pmatrix} 12.58 \\ -1.38 \\ 8.89 \\ -0.09 \end{pmatrix} = \begin{pmatrix} 0.6659 \\ -0.2238 \\ -0.1620 \\ -0.0517 \end{pmatrix} W_2 + \begin{pmatrix} -0.1883 \\ -1.2808 \\ -0.2033 \\ -0.2959 \end{pmatrix} T + \begin{pmatrix} 0.6705 \\ 1.1188 \\ 0.8102 \\ 0.2584 \end{pmatrix} G + \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} z_c^+ + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} z_c^- + \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix} z_p^+ + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} z_p^- + \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} z_{w_1}^+ + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} z_{w_1}^- + \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} z_I^+ + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} z_I^-$$

$$W_2, T, G, z_c^+, z_c^-, z_p^+, z_p^-, z_{w_1}^+, z_{w_1}^-, z_I^+, z_I^- \geqq 0,$$

where (16) and (17) correspond to (5) and (6), and where each goal is given equal weight. An optimal solution to this problem leads to the policy

$$\text{(18)}\qquad x_0 = \begin{pmatrix} 6.870 \\ 13.650 \\ 15.770 \end{pmatrix},$$

and the corresponding target configuration y_0 is

$$\text{(19)}\qquad y_0 = \begin{pmatrix} 50.16 \\ 7.70 \\ 31.90 \\ -5.71 \end{pmatrix}.$$

Comparison of y_0 with the prescribed vector y^* indicates that the three goals C^*, P^*, and W_1^* are attained exactly and the private investment goal I^* is underattained by \$.23 billion.

Suppose that the policy maker, upon being informed of (19), wants to assign a larger weight to the attainment of the investment goal I^* than is the case in the objective (16). Then the deviation variable z_I^- can be given the weight k (z_I^+ being given the same weight 1), and we

$$\text{minimize } z_c^+ + z_c^- + z_p^+ + z_p^- + z_{w_1}^+ + z_{w_1}^- + z_I^+ + kz_I^- \tag{20}$$

subject to (17). We study this optimization for various values of k (continuous parameterization of k). Table 1 summarizes relevant information. For $1 \leqq k < 4.328$ an optimal solution remains unchanged and for each such choice of weighting factor k, x_0 in (18) is optimal and y_0 in (19) is realized. This is indicated by the first data row of Table 1. For weight $k \geqq 4.328$, we must shift to a new optimal solution x_0, getting a corresponding y_0. The second data row in Table 1 displays values of these vectors. The investment goal is attained for weights k in the range indicated and we overattain the goal P^* by one billion.

TABLE 1. *Optimal Policies and Consequent Goal Attainments*

Range of Values of k	Components of y_0				Components of x_0		
	C_0	P_0	$W_{1,0}$	I_0	$W_{2,0}$	T_0	G_0
$1 \leqq k < 4.328$	50.16	7.7	31.9	−5.71	6.870	13.650	15.770
$4.328 \leqq k < \infty$	50.16	8.7	31.9	−5.48	6.845	12.641	15.512

Note: Underlining indicates exact attainment of goal.

Comparison of the two optimal policies indicates that attainment of I^* is brought about by increased deficit spending; $G_0 - T_0 = 15.770 - 13.650 = 2.120$ billion for the original optimal x_0, whereas $G_0 - T_0 = 15.512 - 12.641 = 2.871$ for the alternate optimal x_0. Note that the increased deficit is not attributable to increased values of G_0, but rather to decreased values of T_0. That is, in this model optimal adjustments call for reducing taxes rather than increasing government expenditures.

A policy maker might wish to inform himself further about the interactions between T, G, I, and optimal policies. T, G, and I are related by means of the fourth equation in (15),

$$I = -0.0517W_2 - 0.2959T + 0.2584G - 5.4938.$$

One could study how sensitive an optimal policy is to changes in either of the multipliers −0.2959 and 0.2584. A computer program can be used to parameterize the matrix in (17), one row or one column at a time, so parameterizing any one element in the matrix is a special case. Specifically, one could determine a range of values containing −0.2959 within which an x_0 remains optimal, assuming that other columns of the matrix are held constant. Similar remarks can be made concerning the parameterization of the government expenditures multiplier 0.2584. Indeed, the following kinds of parametric analysis can be performed on the model (16) and (17): (a) nonnegative parameterizing of the coefficients in (16); (b) parameterizing the column vector of constants in (17); (c) parameterizations of (a) and (b) simultaneously; (d) parameterizing the matrix in (17) one row or one column at a time. Thus the model provides the policy maker with considerable flexibility in both policy formulation and decision making.

EXAMPLE 2: Prespecification of Both Goals and Policies: Theil [11] introduced an optimization model in which the policy maker specifies a goal vector, y^*, and a desired instrument vector, x^*. We reformulate this model by means of goal programming and introduce additional flexibilities by using alternate objective functions.

The target or goal variables chosen by the policy maker are C, I, and D, where D is a distribution variable defined as $W_1 - 2P$ (see Theil [11], p. 79). Prespecified goal values are, in billions of 1933 dollars, $C^* = 49.69$, $I^* = -3.10$, and $D^* = 11.25$. The desired instrument values for W_2, T, and G are 5.04, 7.40, and 10.44, respectively (for specification of goal and instrument variables, see Theil [11]). Thus

$$y^{*T} = (C^*, I^*, D^*) = (49.69 \quad -3.10 \quad 11.25) \tag{21}$$

and

$$x^{*T} = (W_2^*, T^*, G^*) = (5.04 \quad 7.40 \quad 10.44). \tag{22}$$

The goal programming model is to

$$\text{minimize } e^T z^+ + e^T z^- \tag{23}$$

subject to

$$\begin{pmatrix} y^* \\ x^* \end{pmatrix} - \begin{pmatrix} s \\ 0 \end{pmatrix} = \begin{pmatrix} R \\ I_3 \end{pmatrix} x - I_6 z^+ + I_6 z^-, \tag{24}$$

$$x, z^+, z^- \geqq 0,$$

where $\begin{pmatrix} y^* \\ x^* \end{pmatrix}$ in (24) comes from (21) and (22), and where

$$s = \begin{pmatrix} 37.5812 \\ -5.4939 \\ 4.8580 \end{pmatrix}, \quad R = \begin{pmatrix} 0.6659 & -0.1833 & 0.6705 \\ -0.0517 & -0.2959 & 0.2584 \\ 0.2855 & 2.3583 & -1.4274 \end{pmatrix}$$

were calculated from data in Klein's model 1 in the manner suggested by Theil (see Theil [11] p. 87 and Tamura [10], p. 46), I_3 and I_6 denote identity matrices of size 3 and 6, respectively, and

$$z^{+T} = (z_C^+, z_I^+, z_D^+, z_{W_2}^+, z_T^+, z_G^+),$$
$$z^{-T} = (z_C^-, z_I^-, z_D^-, z_{W_2}^-, z_T^-, z_G^-),$$
$$x^T = (W_2, T, G).$$

As observed earlier, the objective function we are using assigns the same weights to each z_i^+ and z_i^-. If the policy maker would accept an overattainment of goals C^* and I^* and an underattainment of goal D^*, then minimization of the following objective function subject to (24) could be suggested,

$$f^T \begin{bmatrix} z^+ \\ z^- \end{bmatrix}, \tag{25}$$

where f is a column vector which contains 12 elements all but 3 of which are set equal to 1. The remaining three elements are the coefficients of z_C^+, z_I^+, and z_D^- and are set equal to zero. The zero coefficient of these variables then essentially excludes them from the minimization process.

Optimal policy solutions under the two different objectives are given in Table 2. For purposes of comparison we have also indicated a Theil type optimization decision, i.e., one from a L_2 metric model.

TABLE 2. *Alternate Optimal Policy Decisions (Billions of 1934 Dollars)*

	Values of y			Values of x		
	C	I	D	W_2	T	G
Desired Values	49.69	−3.10	11.25	5.04	7.40	10.44
Optimal Solution with (23)	46.35	−5.55	11.25	5.04	8.42	10.44
Optimal Solution with (25)	46.54	−5.25	8.84	5.04	7.40	10.44
Theil Model	47.77	−5.41	10.42	6.44	8.32	11.13

Note: Underlining indicates exact attainment of goal.

5. CONCLUDING REMARKS

Earlier we indicated a generalization of optimization models in terms of the L_p metric, a generalization which includes both goal programming and Theil type optimization models as special cases, i.e.,

$$\text{minimize} \left(\sum_i |y_i^* - y_i|^p \right)^{1/p},$$

subject to

$$y = Rx + s, \; x \geqq 0.$$

Although for $p \geqq 2$ this is a nonlinear optimization problem and, as such, may appear to present insurmountable computational difficulties, recent research (Duffin, Peterson, and Zener [3] and Peterson and Ecker [8]) has shown that this problem is one of a class of nonlinear optimization problems known as geometric programming. Results for these models are available whereby numerical calculations and approximations can be developed. Moreover, certain types of duality theorems and "complementary slackness" properties have been established, the economic interpretations of which are yet to be explored. Research by the authors on these matters is now in progress.

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PRODUCTION AND EMPLOYMENT SCHEDULING IN MULTISTAGE PRODUCTION SYSTEMS

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ABSTRACT

A model is developed for planning optimal production and employment levels in multiproduct, multistage production systems. The market requirements for each product over the planning period are given. Backorders and/or shortages are permitted. Backorders and shortages must be considered in order to determine the amount of each product's demand that should be filled, backlogged, or go unsatisfied if the production capacity is insufficient to fill all market requirements. Backorders and shortages, on the other hand, are desirable under certain dynamic market conditions.

I. INTRODUCTION

The production and employment scheduling problem—planning production and employment levels over time—has received a great deal of attention. Models and decision-rules generally based on single-stage production systems have been developed for many situations under various conditions [7]. Whenever many products are to be manufactured the concept of aggregate production scheduling has frequently been suggested [1, 5]. The aggregate production level is then represented by some appropriate measure of production per unit of time [5] in view of the common denominator required. Once the aggregate production level is specified the problem of determining the quantities and the timing for the individual products still remains. This problem of disaggregation is of special importance whenever the available capacity is insufficient to produce all market requirements and therefore some demand has to go unsatisfied. It is not at all obvious how the disaggregation decisions should be made in view of insufficient capacity.

Following along the lines of more recent work in production planning [2, 3, 8], we consider a multistage production system consisting of various production facilities (stages) linked together by in-process inventories. The objective of this paper is to develop a model for multiproduct, multistage production systems by which optimal production decisions on a disaggregate basis for both sufficient and insufficient production capacity can be made. As a point of departure we take Zangwill's [8] model. The subsequent analysis, however, differs from Zangwill's work on a number of points:

(a) The demand requirements need not be met.
(b) Each production stage manufactures various different products.
(c) Workforce decisions are integrated.
(d) A different cost structure is assumed.

II. THE MODEL

The Production System

The elements of a multistage production system are the individual production stages consisting of production facilities and an inventory. These production facilities can be thought of as multipurpose

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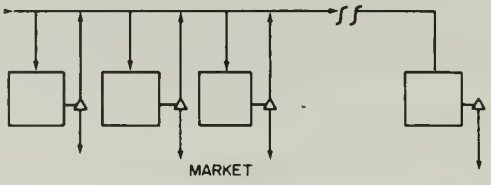
machinery. The output of a given production stage will be fed into the inventory bank from which the units can be used either as inputs to higher numbered production stages or to supply market requirements for the stage's products. Figure 1 describes such a production system.



Production and Inventory Aspects

Let D_{jt}^{i} be the market requirements of production stage j's ($j=1, 2, \ldots, J$) product i ($i=1, 2, \ldots, I$) during the period t($t=1, 2, \ldots, T$). It is assumed that the requirements in each period over the planning period are known in advance. The planning period is medium range so no discounting is required. The market requirements are expressed in units. Note that the index i at each production stage represents physically different products. Let H_{jt}^{i} be the amount of product i in the inventory of stage j at the end of period t. The inventory equation can then be expressed by (1).

$$(1) \qquad H_{jt}^{i}=H_{jt-1}^{i}+X_{jtR}^{i}+X_{jtO}^{i}-D_{jt}^{i}-\sum_{k=j+1}^{J}\sum_{n=1}^{I}\alpha_{jk}^{in}\,(X_{ktR}^{n}+X_{ktO}^{n}) \qquad \text{for all } i,\ t \text{ and } j=1, \ldots, J-1$$

with X_{jtR}^{i} (X_{jtO}^{i}) being the amount of product i produced in stage j during period t on regular time (overtime). α_{jk}^{in} ($\alpha_{jk}^{in}\geqslant O$) can be considered as a production coefficient and is the number of units of production stage j's product i required to produce one unit of stage k's ($k=j+1, j+2, \ldots, J$) product n ($n=1, 2, \ldots, I$). $\alpha_{jk}^{in}\equiv O$ if product i of stage j is not being used for the production of stage k's product n. The inventory equation for the last stage J of the production system is

$$(2) \qquad H_{Jt}^{i}=H_{Jt-1}^{i}+X_{JtR}^{i}+X_{JtO}^{i}-D_{Jt}^{i} \qquad \text{for all } i,\ t.$$

The formulation of (1) and (2) assumes that the production capacity available is sufficient to satisfy all market requirements. If market requirements exceed the available production capacity, the excess demand must either be backlogged into the next period or has to go unsatisfied. The possibility for backlogs and shortages is only meaningful for market requirements since the requirements of subsequent production stages, i.e., the production levels, are variable themselves. The inventory equation, adjusted by backlogs and shortages, can then be expressed by (3)

$$(3) \qquad H_{jt}^{i}=H_{jt-1}^{i}+X_{jtR}^{i}+X_{jtO}^{i}-(D_{jt}^{i}+B_{jt-1}^{i})+B_{jt}^{i}+S_{jt}^{i}-\sum_{k=j+1}^{J}\sum_{n=1}^{I}\alpha_{jk}^{in}\,(X_{ktR}^{n}+X_{ktO}^{n}$$

$$\text{for all } i,\ t \text{ and } j=(j+1, \ldots J-1)$$

with B_{jt}^{i} being the demand backlogged into period $t+1$ and S_{jt}^{i} the amount of the periods total demand ($D_{jt}^{i}+B_{jt-1}^{i}$) which will never be filled. Note that the variables H_{jt}^{i}, B_{jt}^{i}, and S_{jt}^{i} are nonnegative variables. Generally, one can assume that, whenever a positive inventory exists of production stage j's product i at the end of period t, B_{jt}^{i} and S_{jt}^{i} will be zero; on the other hand, whenever B_{jt}^{i} and/or S_{jt}^{i} is greater than zero, there will normally be no inventory available. It should be realized, however, that the formulation of (3) permits the possibility of having positive inventory as well as backlogged and/or unfilled demand at the same time. This configuration can be desirable considering over time changing

economic conditions. For example, in view of a speculative increase in the selling price in periods ahead, it might be advantageous to let some of the current periods demands go unfilled, though inventory is available. The inventory equation for the last production stage is

$$H^i_{Jt} = H^i_{Jt-1} + X^i_{JtR} + X^i_{JtO} - (D^i_{Jt} + B^i_{Jt-1}) + B^i_{Jt} + S^i_{Jt}, \qquad \text{for all } i, t. \tag{4}$$

For the special case of a production line (i.e., production stage j's output will only be used in production stage $j+1$) we have instead of (3),

$$H^i_{jt} = H^i_{jt-1} + X^i_{jtR} - X^i_{jtO} - (D^i_{jt} + B^i_{jt-1}) + B^i_{jt} + S^i_{jt} - \sum_{n=1}^{I} \alpha^{in}_{jj+1} (X^n_{j+1tR} + X^n_{j+1tO}), \tag{3a}$$

for all i, t.

In view of limited in-process inventory capacities, we can restrict

$$\sum_{i=1}^{I} \beta^i_j H^i_{jt} \leq L_{jt} \qquad \text{for all } j, t, \tag{5}$$

where β^i_j is the amount of space required to store one unit of production stage j's product i. L_{jt} represents the available inventory capacity in space terms. Correspondingly, restrictions can be placed on the maximum amount to be backlogged per period. Such an upper limit is desirable for B^i_{jT} in view of the fact that the cost associated with backlogs are normally less than those associated with shortages. The production quantities are subject to the following capacity restrictions

$$\sum_{i=1}^{I} \tau^i_j X^i_{jtR} \leq R_{jt} \qquad \text{for all } j, t \tag{6}$$

and

$$\sum_{i=1}^{I} \tau^i_j X^i_{jtO} \leq O_{jt} \qquad \text{for all } j, t, \tag{7}$$

where τ^i_j is the time required to produce one unit of production stage j's product i and R_{jt} (O_{jt}), the capacity in time units of regular time (overtime) available on stage j during period t.

Workforce Considerations

The production decisions can only be carried out if the necessary workforce is available. The workforce decisions are to be made in view of the production requirements and with respect to the dynamic costs involved. (For a discussion of dynamic costs see [5]). The workforce available during period t will be denoted by W_t. According to [6] we can differentiate between "productive" (W^P_t) and "idle" workforce (W^I_t). So

$$W_t = W^P_t + W^I_t.$$

This distinction is desirable in view of the possibility of keeping workers on a reduced pay rate rather than laying them off. Let γ^i_j be the number of units of stage j's product i which can be produced by one worker during period t. Assuming a constant worker productivity (i.e., a constant γ^i_j), the required productive workforce for regular time or overtime production in t can be expressed by

$$W^P_{tR} = \sum_{j=1}^{J} \sum_{i=1}^{I} \frac{1}{\gamma^i_j} X^i_{jtR} \tag{9}$$

and

$$W_{tO}^{P} = \sum_{j=1}^{J} \sum_{i=1}^{I} \frac{1}{\gamma_j^i} X_{jtO}^i. \tag{10}$$

The productive workforce in period t is therefore

$$W_t^P = \sum_{j=1}^{J} \sum_{i=1}^{I} \frac{1}{\gamma_j^i} (X_{jtR}^i + X_{jtO}^i). \tag{11}$$

The total workforce available in period t can then be expressed by

$$\sum_{j=1}^{J} \sum_{i=1}^{I} \frac{1}{\gamma_j^i} (X_{jtR}^i + X_{jtO}^i) + W_t^I = \sum_{j=1}^{J} \sum_{i=1}^{I} \frac{1}{\gamma_j^i} (X_{jt-1R}^i + X_{jt-1O}^i) + W_{t-1}^I + G_{t-1} - F_{t-1} \qquad \text{for all } t, \tag{12}$$

with G_{t-1} being the number of workers hired at the end of period $t-1$ and F_{t-1} the number of workers laid off at the end of $t-1$. Since the decision of which workers to lay off is usually made on grounds of seniority, we don't have to specifically account for this fact within the model. The same holds for the "idle" workforce which is generally determined by juniority. The number of workers to be assigned to the various production stages during period t can easily be obtained from (9) and (10).

From an organizational point of view, it is necessary to maintain the proper relation between the number of processing workers and supervisory labor. Let W_t^S be the number of supervisory labor in period t, then we have an analogous situation to (12)

$$W_t^S = W_{t-1}^S + M_{t-1} - N_{t-1} \qquad \text{for all } t, \tag{13}$$

with M_{t-1} being the number of supervisory labor added and N_{t-1} the number laid off at the end of period $t-1$. The relation between the processing and supervisory labor can then be expressed as

$$\frac{1}{\delta} \sum_{j=1}^{J} \sum_{i=1}^{I} \frac{1}{\gamma_j^i} (X_{jtR}^i + X_{jtO}^i) \leq W_t^S, \tag{14}$$

with δ being the span of control.

The Objective Function

The objective of the model is to determine the production and employment levels so that the contribution to profits over the planning period is maximized.

Let r_{jt}^i be the selling price of one unit of production stage j's product i during t. If we assume that r_{jt}^i is constant for all t, the total revenues of the production system are

$$\sum_{t=1}^{T} \sum_{j=1}^{J} \sum_{i=1}^{I} r_j^i (D_{jt}^i - S_{jt}^i),$$

since any amount backlogged into a future period will bring the same revenue. Assuming an over time changing selling price r_{jt}^i, the total revenue depends on the actual price any backlogged demand can ultimately be sold for. If backlogged demand B_{jt}^i has to be filled at the existing price r_{jt}^i of period t the total revenues are

$$\sum_{t=1}^{T} \sum_{j=1}^{J} \sum_{i=1}^{I} r_{jt}^i (D_{jt}^i - S_{jt}^i).$$

If on the other hand backlogged demand will be filled at current prices, the total revenues become

$$\sum_{t=1}^{T}\sum_{j=1}^{J}\sum_{i=1}^{I} r_{jt}^{i}\,(D_{jt}^{i}+B_{jt-1}^{i}-B_{jt}^{i}-S_{jt}^{i}).$$

Production costs are the sum of material and labor costs. Capital cost will be considered as fixed. The cost to produce stage j's product i depends not only on the decisions made on stage j but also on the production and employment decisions made on the lower-numbered stages which provide input to stage j. However, if one handles the labor costs *en bloc*, the material costs of stage j's product i are only the costs for additional raw material since the costs of the inputs from lower numbered stages are already accounted for. Let c_{jt}^{i} be the cost of additional raw material to produce one unit of stage j's product i during period t. The total material costs are then

$$\sum_{t=1}^{T}\sum_{j=1}^{J}\sum_{i=1}^{I} c_{jt}^{i}\,(X_{tjR}^{i}+X_{tjO}^{i}).$$

The inventory costs are the product of the average amount in inventory and an inventory cost factor z_{j}^{i}. Therefore, we have

$$\sum_{t=1}^{T}\sum_{j=1}^{J}\sum_{i=1}^{I}\frac{1}{2}\,z_{j}^{i}\,(H_{jt-1}^{i}+H_{jt}^{i}).$$

Backlog and shortage costs are

$$\sum_{t=1}^{T}\sum_{j=1}^{J}\sum_{i=1}^{I}(b_{j}^{i}B_{jt}^{i}+s_{j}^{i}S_{jt}^{i}),$$

with b_{j}^{i} being the penalty for backlogging one demand unit of stage j's product i for one period. s_{j}^{i} is is the expected cost in terms of loss of goodwill, over and above the lost revenue, of one unit of stage j's product i that cannot be filled.

The costs associated with the labor-force are

$$\sum_{t=1}^{T}\sum_{j=1}^{J}\sum_{i=1}^{I}\frac{1}{\gamma_{j}^{i}}\,[p(X_{jtR}^{i})+p(1+\kappa)X_{jtO}^{i}]+p(1-\lambda)W_{t}^{I}+(h\cdot G_{t}+f\cdot F_{t})+p_{S}\cdot W_{t}^{S}+h_{S}\cdot M_{t}+f_{S}\cdot N_{t},$$

with p being the regular-time pay rate of a processing worker per period, κ the overtime premium, and λ the pay reduction for the idle workforce. $h(h_S)$ is the hiring cost per worker (supervisor), $f(f_S)$ the cost of laying off one worker (supervisor), and p_S the supervisor's pay rate per period. Combining the above expressions* leads to the objective function (15)

(15)
$$TP=\sum_{t=1}^{T}\sum_{j=1}^{J}\sum_{i=1}^{I}\{r_{j}^{i}(D_{jt}^{i}-S_{jt}^{i})-c_{j}^{i}(X_{jtR}^{i}+X_{jtO}^{i})$$

$$-\frac{1}{2}\,z_{j}^{i}(H_{jt-1}^{i}+H_{jt}^{i})-b_{j}^{i}B_{jt}^{i}-s_{j}^{i}S_{jt}^{i}$$

$$-\frac{1}{\gamma_{j}^{i}}\,[pX_{jtR}^{i}+p(1+\kappa)X_{jtO}^{i}]-p(1-\lambda)W_{t}^{I}$$

*Assuming an overtime constant price r_{jt}^{i}.

$$-hG_t - fF_t - p_S W_t^S - h_S G_t^S - f_S F_t^S\}$$

$$= \sum_{t=1}^{T} \sum_{j=1}^{J} \sum_{i=1}^{I} \left\{ r_j^i D_{jt}^i - \left(c_{jt}^i + \frac{p}{\gamma_j^i} \right) X_{jtR}^i - \left(c_{jt}^i + \frac{p(1+\kappa)}{\gamma_j^i} \right) X_{jtO}^i \right.$$

$$-\frac{1}{2} z_j^i (H_{jt-1}^i + H_{jt}^i) - b_j^i B_{jt}^i - (r_j^i + s_j^i) S_{jt}^i$$

$$\left. -p(1-\lambda) W_t^I - h \cdot G_t - f \cdot F_t - p_S W_t^S - h_S M_t - f_S N_t \right\}.$$

Maximizing (15) with respect to (3) or (3a), (4), (5), (6), (7), (12), (13), (14), and the nonnegativity constraints for the variables is a problem that can be solved by linear programming.†

III. CONCLUSION

The purpose of model was to formulate the production and employment scheduling problem on a disaggregate basis in multiproduct, multistage production systems. The model determines the amount of the demand for each product that should be satisfied, be backlogged, or remain unfilled. These results are important whenever the production capacity is insufficient to produce all market requirements. On the other hand, the formulation can be very useful under certain dynamic market conditions. The model determines also how the productive workforce is to be assigned to the various production stages. The model seems to be applicable to integrated iron and steel manufacturing systems as well as to various chemical processes.

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†Note that $\sum_t \sum_j \sum_i r_j^i D_{jt}^i$ is a constant.

A MUTUAL PRIMAL-DUAL LINEAR PROGRAMMING ALGORITHM

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ABSTRACT

A new primal-dual linear programming algorithm is exhibited. A proof is given that optimal solutions to both primal and dual problems (when such solutions exist) are found in a finite number of steps by this algorithm. A numerical example is included to illustrate the method.

1.0 INTRODUCTION

Since the publication of the primal-dual algorithm by Dantzig, Ford, and Fulkerson, in 1956 (see Refs. [3], [5], [7]), apparently little work has been done in this area (see Gomory and Balinski [1]). This paper presents a primal-dual algorithm which is similar to that of Dantzig, Ford, and Fulkerson in that it uses the complementary slackness concept and alternates between primal and dual problems; however, it differs in that it is simpler, but requires a separate Phase I stage.*

Essentially, this algorithm consists in generating and solving a sequence of subproblems which use the constraints of the original problem. The objective function at each stage, however, is determined by the solution of the problem of the previous stage. Whether this algorithm is any more efficient than, say, the revised simplex algorithm remains to be seen.†

Before exhibiting the algorithm, we first establish some notational conventions and state some necessary results.

2.0 NOTATION AND BASIC THEOREMS

For the sake of simplicity, and without any loss of essential generality, we will consider only problems of the form

$$\text{maximize } z = c'x$$

subject to

$$\begin{aligned} Ax &\leqslant b \\ x &\geqslant 0 \end{aligned} \tag{2.1}$$

where A is an (nxm) matrix, b is an $(nx1)$ vector with nonnegative entries, c is an $(mx1)$ vector of criterion function coefficients, and x is an $(mx1)$ vector of variables. The "primes" attached to symbols denote the transpose of the indicated matrix. All entries in A, b, and c are taken to be given constants.

We may state the dual to this problem as

$$\text{minimize } u = w'b \tag{2.2}$$

*For an explanation of the two stages of the simplex method, see Dantzig [4], ch. 5.

†A Fortran program has been written for the CDC 3800 computer to compare the efficiency of this algorithm with the simplex method, but as yet, no conclusive results have been obtained.

subject to

$$w'A \geq c'$$
$$w' \geq 0 \tag{2.3}$$

where w is an $(nx1)$ vector of variables.

These two problems may be rewritten by introducing slack variables s and v for the primal and dual, respectively, and taking the transpose of 2.3 as follows:

maximize $z=c'x$		minimize $u=w'b$	
subject to		subject to	
	$Ax+Is=b$		$A'w-Iv=c$
(2.4)	$(x, s) \geq 0$	(2.5)	$(w, v) \geq 0,$

where I is an appropriately dimensioned identity matrix in each case.

We will digress here slightly to define formally the notion of complementary slackness, a concept which is central to the algorithm.

DEFINITION: For given basic feasible solutions (x, s) to the primal and (w, v) to the dual, *complementary slackness is satisfied* if, and only if,

$$\text{i) } w \cdot s = 0$$

and

$$\text{ii) } x \cdot v = 0.$$

We conclude this section by stating four well-known results which are essential to our algorithm. Proofs are given by Dantzig in [4].

Suppose we have any two basic feasible solutions (x, s) for the primal (2.4) and (w, v) for the dual (2.5). Then

LEMMA 1: $w'b \geq c'x$.

LEMMA 2: (x, s) and (w, v) are optimal solutions for the primal and dual, respectively, if, and only if, $w'b=c'x$.

THEOREM 1: (x, s) and (w, v) are optimal solutions for the primal and dual, respectively, if, and only if, complementary slackness is satisfied for these solutions.

FUNDAMENTAL DUALITY THEOREM: If a feasible solution to 2.4 exists and z of 2.4 is bounded above, then optimal feasible solutions exist for both the primal (2.4) and the dual (2.5).

3.0 THE ALGORITHM

The algorithm is initiated with basic feasible solutions to both the primal and dual, either or both of which may be contained in the intital formulation of the problem or may be generated by Phase I of the simplex algorithm. We also assume that z of 2.4 has a finite upper bound. The fundamental duality theorem assures us that optimal solutions for both the primal and dual exist.

Recall the primal and dual constraints may be written as

$$Ax+Is=b \text{ (primal)} \tag{3.1}$$

and

$$w'A-v'I=c' \text{ (dual).} \tag{3.2}$$

Multiplying 3.1 by w' on the left and 3.2 by x on the right and applying Lemma 1 gives

$$w \cdot s + v \cdot x = w \cdot b - c \cdot x \geq 0. \tag{3.3}$$

Let $\alpha = w \cdot s + v \cdot x$. Now any decrease in the value of α brings us closer to the optimum since a decrease in α represents a movement toward the satisfaction of complementary slackness and a convergence of $w \cdot b$ and $c \cdot x$ toward their common optimal value (see Lemma 2 and Theorem 1).

We now proceed to define the algorithm. Suppose we have a basic feasible solution (x^0, s^0) to the primal and a basic feasible solution (w^0, v^0) to the dual with

$$(x^0, s^0) \cdot (v^0, w^0) = x^0 \cdot v^0 + s^0 \cdot w^0 > 0$$

(i.e., the solutions are non-optimal). Let Y^0 be a current full-tableau matrix for the primal, Y_0^0 be a current constant column for the primal, Z^1 be a current full-tableau matrix for the dual, and Z_0^1 be a current constant column for the dual. Consider the linear programming problem

$$\text{minimize } \alpha_0 = (x, s) \cdot (v^0, w^0) \tag{3.4}$$

subject to

$$Y^0(x, s) = Y_0^0 \tag{3.5}$$

$$(x, s) \geq 0.$$

Suppose the optimal (if one exists) solution to 3.4 and 3.5 is (x^1, s^1) and

$$(x^1, s^1) \cdot (v^0, w^0) = \alpha_0^*.$$

Let Y^1 and Y_0^1 be the matrix and constant column, respectively, corresponding to this solution. If $\alpha_0^* = 0$, then (x^1, s^1) and (w^0, v^0) are optimal solutions to the primal and dual, respectively, by Theorem 1. If $\alpha_0^* > 0$ (recall from 3.3 that $\alpha_0^* \geq 0$), consider the problem

$$\text{minimize } \alpha_1 = (x^1, s^1) \cdot (v, w) \tag{3.6}$$

subject to

$$Z^1(w, v) = Z_0^1 \tag{3.7}$$

$$(w, v) \geq 0.$$

Let the optimal tableau to this problem be represented by Z^2 and Z_0^2 and the optimal solution be (w^1, v^1); let

$$(x^1, s^1) \cdot (v^1, w^1) = \alpha_1^*.$$

As before, if $\alpha_1^* = 0$, we are through. If not, we return to the primal and solve

$$\text{minimize } \alpha_2 = (x, s) \cdot (v^1, w^1) \tag{3.8}$$

subject to

$$Y^1(x, s) = Y_0^1 \tag{3.9}$$

$$(x, s) \geq 0$$

for (x^2, s^2), Y^2, Y_0^2, and $\alpha_2^* = (x^2, s^2) \cdot (v^1, w^1)$. The algorithm consists in following this procedure of using the optimal solution at each stage as the criterion function coefficients for the next stage until for some n, $\alpha_n^* = 0$.

In the next section we will show that the algorithm results in optimal solutions for both the primal and dual, whenever they exist, in a finite number of iterations.

4.0 PROOF OF CONVERGENCE

We may state the subproblems described above in general as follows:

Primal

$$\text{minimize } \alpha_k = (x, s) \cdot (v^{[k]}, w^{[k]})$$

subject to

$$\begin{aligned} Y^{[k]}(x, s) &= Y_0^{[k]} \\ (x, s) &\geq 0, \end{aligned} \tag{4.1}$$

where $[k] = k/2$ for $k = 0, 2, 4, \ldots$ and

Dual

$$\text{minimize } \alpha_m = (x^{[m]}, s^{[m]}) \cdot (v, w)$$

subject to

$$\begin{aligned} Z^{[m]}(w, v) &= Z_0^{[m]} \\ (w, v) &\geq 0, \end{aligned} \tag{4.2}$$

where $[m] = (m-1)/2 + 1$ for $m = 1, 3, 5, \ldots$ We first show that the subproblems 4.1 and 4.2 have optimal solutions.

THEOREM 2: Both problems (4.1 and 4.2 above) have optimal solutions for any $k \epsilon \{0, 2, 4, \ldots\}$ and any $m \epsilon \{1, 3, 5, \ldots\}$.

PROOF: Both problems (4.1 and 4.2) have feasible solutions by assumption. Now for every basic feasible solution (x, s) to the primal and (w, v) to the dual, we have by Lemma 1

$$x \cdot v + w \cdot s \geq 0. \tag{4.3}$$

Since the problems in question are both minimizing problems whose objective functions are of the form of 4.3, we have just seen that both are bounded below by zero. By the fundamental duality theorem then, both have optimal solutions.

Finally, we show that the algorithm does indeed converge.

THEOREM 3: $\alpha_n^* = 0$ for some (finite) n.

PROOF: We assume that the original problem has an optimal solution. Clearly

$$\alpha_1^* \leq \alpha_0^*,$$

since at worst (w^0, v^0) satisfies 3.7 and $(x^1, s^1) \cdot (v^0, w^0) = \alpha_0^*$ (this fact is just a general property of constrained optima, that is, that the optimal solution to a minimizing problem is no larger than any admissable solution). Suppose $\alpha_1^* = \alpha_0^*$. Then for every basic feasible solution (w, v) for the dual

$$\begin{aligned} w \cdot b - c \cdot x^1 &= w \cdot s^1 + s^1 + x^1 \cdot v \qquad \text{(by 3.3)} \\ &= \alpha_1 \geq \alpha_1^* = \alpha_0^* \qquad \text{(assumption)} \\ &= w^0 \cdot s^1 + x^1 \cdot v^0 \\ &= w^0 \cdot b - c \cdot x^1 \qquad \text{(by 3.3).} \end{aligned}$$

Thus for every basic feasible solution (w, v) for the dual

$$w \cdot b - c \cdot x^1 \geq w^0 \cdot b - c \cdot x^1 \text{ or } w \cdot b \geq w^0 \cdot b,$$

which implies that (w^0, v^0) is the optimal solution for the dual. Similarly, for every basic feasible solution (x, s) for the primal

$$\begin{aligned} w^0 \cdot b - c \cdot x &= w^0 \cdot s + x \cdot v^0 && \text{(by 3.3)} \\ &= \alpha_1 \geq \alpha_1^* = \alpha_0^* && \text{(assumption)} \\ &= w^0 \cdot s^1 + x^1 \cdot v^0 \\ &= w^0 \cdot b - c \cdot x^1 && \text{(by 3.3)} \end{aligned}$$

Thus for every basic feasible solution (x, s) for the primal

$$w^0 \cdot b - c \cdot x \geq w^0 \cdot b - c \cdot x^1 \text{ or } c \cdot x \leq c \cdot x^1$$

which implies that (x^1, s^1) is optimal for the primal. Now if $\alpha_0^* > 0$, then complementary slackness is not satisfied by the solutions (x^1, s^1) and (w^0, v^0), and so at least one must be non-optimal by Theorem 1. This contradiction (i.e., we just showed that if $\alpha_1^* = \alpha_0^*$, both (w^0, v^0) and (x^1, s^1) are optimal) implies that if $\alpha_0^* > 0$, then

$$\alpha_1^* < \alpha_0^*.$$

We may repeat this argument for α_1 and α_2 and so on. We conclude therefore that

$$\alpha_k^* > 0 \text{ implies that } a_{k+1}^* < \alpha_k^* \text{ for each } k = 0, 1, 2, \ldots.$$

Since there are only a finite number of basic feasible solutions to the primal and dual, and since each iteration of the algorithm results in a strict decrease in α, if optimal solutions exist, then by Theorem 1, there exists an n (finite) such that

$$\alpha_n^* = 0.$$

Furthermore the solution to the problem for which α_n is the objective function is the optimal solution to either the primal or dual with the solution to the preceeding problem being the optimal solution to the other.

5.0 NUMERICAL EXAMPLE

Consider the following primal problem (this problem was taken from Hadley [5], p. 269):

$$\text{maximize } z = 2x_1 + 13x_2 + 7x_3 + 6x_4$$

subject to

$$\begin{aligned} 1 &\geq 4x_1 + 1x_2 + 3x_3 + 5x_4 \\ 3 &\geq -x_1 + 3x_2 + 3x_3 - 3x_4 \\ 7 &\geq 2x_1 - 4x_2 + 1x_3 + 8x_4 \\ 5 &\geq 3x_1 + 5x_2 - 2x_3 - 4x_4 \\ 2 &\geq -5x_1 - 1x_2 + 1x_3 + 6x_4 \\ 8 &\geq 6x_1 + 2x_2 - 9x_3 + 1x_4 \end{aligned} \tag{5.1}$$

The dual is then

$$\text{minimize } u = w_1 + 3w_2 + 7w_3 + 5w_4 + 2w_5 + 8w_6$$

subject to

$$
\begin{aligned}
2 &\leq 4w_1 - 1w_2 + 2w_3 + 3w_4 - 5w_5 + 6w_6 \\
13 &\leq 1w_1 + 3w_2 - 4w_3 + 5w_4 - 1w_5 + 2w_6 \\
7 &\leq 3w_1 + 7w_2 + 1w_3 - 2w_4 + 1w_5 - 9w_6 \\
6 &\leq 5w_1 - 3w_3 + 8w_3 - 4w_4 + 6w_5 + 1w_6
\end{aligned} \tag{5.2}
$$

After appending slack variables s_1 through s_6 to 5.1 and v_1 through v_4 to 5.2 and writing both systems in detached coefficient form, we have

Primal

b	x_1	x_2	x_3	x_4	s_1	s_2	s_3	s_4	s_5	s_6
0	2	13	7	6	0	0	0	0	0	0
1	4	1	3	5	1	0	0	0	0	0
3	−1	3	7	−3	0	1	0	0	0	0
7	2	−4	1	8	0	0	1	0	0	0
5	3	5	−2	−4	0	0	0	1	0	0
2	−5	−1	1	6	0	0	0	0	1	0
8	6	2	−9	1	0	0	0	0	0	1

(5.3)

Dual

c	w_1	w_2	w_3	w_4	w_5	w_6	v_1	v_2	v_3	v_4
0	1	3	7	5	2	8	0	0	0	0
2	4	−1	2	3	−5	6	−1	0	0	0
13	1	3	−4	5	−1	2	0	−1	0	0
7	3	7	1	−2	1	−9	0	0	−1	0
6	5	−3	8	−4	6	1	0	0	0	−1

(5.4)

Carrying out a phase I step of the simplex algorithm on 5.4 gives us the dual tableau (see Table 5.5) in basic feasible form (the objective row has been omitted since it is not needed in the rest of the procedure).

Now we construct an objective row for 5.3 from the basic feasible solution to the dual (5.5). This is shown in Table 5.6.

After adjoining 5.6 to 5.3, four pivots of the simplex algorithm are required to minimize. The result is shown in Table 5.7.

Since the objective function has not been reduced to zero (entry (1, 1) in 5.7), at least one dual cycle is required. We construct an objective row for the dual (5.5) from the solution to the primal (5.7). This is shown in Table 5.8.

After adjoining 5.8 to 5.5, one pivot of the simplex algorithm is required to minimize. The result is given in Table 5.9.

Since the value of the objective function in 5.9 is now zero, 5.9 must be the optimal dual tableau (except for the objective row, of course). Thus the previous primal tableau (5.7) must have been optimal. Checking the values of the actual objective functions using the solutions given in 5.7 and 5.9 for the primal and dual, respectively, we find that both are 13.

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TABLE 5.5

c	w_1	w_2	w_3	w_4	w_5	w_6	v_1	v_2	v_3	v_4
1.2979	1.0000	0.0000	0.8511	0.0000	0.0000	0.3830	−0.0164	−0.0213	−0.0426	−0.0851
2.2098	0.0000	0.0000	−0.7328	1.0000	0.0000	1.0040	0.0247	−0.2044	0.0797	−0.0267
0.8179	0.0000	1.0000	−0.4206	0.0000	0.0000	−1.1544	0.0351	−0.0302	−0.1069	0.0421
1.8006	0.0000	0.0000	−0.0747	0.0000	1.0000	−0.0604	0.1227	−0.1336	0.0351	−0.0925

TABLE 5.6

b	x_1	x_2	x_3	x_4	s_1	s_2	s_3	s_4	s_5	s_6
18.4018	0.0000	0.0000	0.0000	0.0000	1.2979	0.8179	0.0000	2.2098	1.8006	0.0000

TABLE 5.7

b	x_1	x_2	x_3	x_4	s_1	s_2	s_3	s_4	s_5	s_6
5.4018	0.0000	0.0000	−51.2537	0.0000	−3.4789	−6.2958	0.0000	1.8733	0.0000	0.0000
0.0000	1.0000	0.0000	−3.4930	0.0000	−0.0423	−0.4085	0.0000	0.2535	0.0000	0.0000
0.0000	0.0000	0.0000	2.6338	1.0000	0.1972	0.2394	0.0000	−0.1831	0.0000	0.0000
11.0000	0.0000	0.0000	2.1268	0.0000	−0.7606	0.6479	1.0000	0.5634	0.0000	0.0000
1.0000	0.0000	1.0000	3.8028	0.0000	0.1831	0.4366	0.0000	−0.0986	0.0000	0.0000
3.0000	0.0000	0.0000	−28.4648	0.0000	−1.2113	−3.0423	0.0000	2.2676	1.0000	0.0000
6.0000	0.0000	0.0000	1.7183	0.0000	−0.3099	1.3380	0.0000	−1.1408	0.0000	1.0000

TABLE 5.8

c	w_1	w_2	w_3	w_4	w_5	w_6	v_1	v_2	v_3	v_4
5.4018	0.0000	0.0000	11.0000	0.0000	3.0000	6.0000	0.0000	1.0000	0.0000	0.0000

TABLE 5.9

c	w_1	w_2	w_3	w_4	w_5	w_6	v_1	v_2	v_3	v_4
0.0000	0.0000	0.0000	−11.0000	0.0000	−3.0000	−6.0000	0.0000	−1.0000	0.0000	0.0000
2.8593	1.0000	0.0000	0.7863	0.0000	0.8672	0.3306	0.0000	−0.1372	−0.0122	−0.1653
1.8473	0.0000	0.0000	−0.7178	1.0000	−0.2013	1.0162	0.0000	−0.1775	0.0726	−0.0081
0.3028	0.0000	1.0000	−0.3992	0.0000	−0.2861	−1.1371	0.0000	0.0080	−0.1169	0.0686
14.6748	0.0000	0.0000	−0.6088	0.0000	8.1500	−0.4923	1.0000	−1.0888	0.2861	−0.7539

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* * *

INTERCEPTION IN A NETWORK*

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ABSTRACT

This paper presents an algorithm for determining where to place intercepting units in order to maximize the probability of preventing an opposing force from proceeding from one particular node in an undirected network to another.

The usual gaming assumptions are invoked; namely, the strategy for placing the units is known to the opponent and he will choose a path through the network which, based on this knowledge, maximizes his probability of successful traverse. As given quantities, the model requires a list of the arcs and nodes of the network, the number of intercepting units available to stop the opposing force, and the probabilities for stopping the opposition at the arcs and nodes as functions of the number of intercepting units placed there. From these quantities, the algorithm calculates the probabilities for placing the unit at the arcs and nodes when one intercepting unit is available, and the expected numbers of units to place at the arcs and nodes when multiple intercepting units are available.

I. INTRODUCTION

The paper depicts a game between two opposing forces—an evader and an interceptor. Only the interceptor's problem will be dealt with. In this game, the evader attempts to proceed from one point to another in a network. The interceptor, who may possess one or more indivisible intercepting units, attempts to stop him by placing these units along arcs or nodes that he expects the evader to traverse. His problem is to place these so as to minimize the evader's probability of successful traverse. The evader's problem is, of course, to select a path that will maximize this probability. Both problems can be represented by a zero-sum two-person game. As the next section will show, however, the game matrix is very large and difficult to generate. By using a steepest ascent approach, a solution may be obtained much more easily for the interceptor.

This paper presents an algorithm based on such an approach. As inputs, it requires a list of the arcs and nodes of the network, the number of intercepting units available to stop the opposing force, and the probabilities for stopping the opposition at the arcs and nodes as functions of the number (integer) of intercepting units placed there. From this information, the algorithm calculates the expected number of intercepting units to place at each arc and each node. For the case where only one intercepting unit is available, the expected values are probabilities of placement; for more than one intercepting unit, expectations are to be interpreted in the most obvious way. Specifically, if the expected number of units to place at a particular location is 4-1/4, one would always place four there and would place a fifth one there one-fourth of the time.

The situation depicted by this algorithm is particularly relevant in applying interdiction to infiltration and counterinsurgency, where the counterinsurgent or infiltrating force would play the role of the

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evader, while the interdiction forces would play that of the interceptor. It also applies to police networks where a wanted fugitive (evader) is attempting to escape from a city while police (intercepting) units are attempting to close off his avenues of escape.

The algorithm to be presented is a steepest ascent approach and yields solutions that are always optimal in the gaming sense (i.e., the evader's best chance of successful traverse is as small as possible) for the interceptor when he has but one intercepting unit at his disposal. For the case of multiple intercepting units, the solutions are not always optimal; thus for the general case the algorithm is an approximation scheme.

II. INTERCEPTOR'S PROBLEM FOR ONE INTERCEPTING UNIT

The network is characterized by sets of elements called arcs and nodes. Nodes are points or junctions, and arcs are undirected line segments joining nodes. An arc that joins node i to node j may be disignated by the symbol (i, j) or (j, i). The node from which the evader starts is called the source; the one he attempts to reach is called the sink.

It is assumed that the evader and interceptor are each composed of one indivisible unit. Later on this restriction will be relaxed for the interceptor. The evader is attempting to choose a path from source to sink in such a way that his probability of successful traverse is maximized. The interceptor wishes to place his intercepting unit so that this probability is minimized. As is customary in game theory, it is assumed that each side's strategy will become known to the other and consequently each will need to adopt a mixed strategy. The inadequacy of pure strategies for the interceptor may be demonstrated by any network whose source and sink are relatively invulnerable to attack and any particular intermediate arc or node may be bypassed. Placing the intercepting unit at the source or sink would be relatively ineffective while placing it anywhere else would be completely ineffective as the evader would choose a path that bypassed the unit.

Quantities defining the local effectiveness of placing the intercepting unit at arcs and nodes, and quantities expressing the interceptor's strategy are as follows:

$p(i)$ = probability that the evader will be stopped at node i, given that node i is on his chosen path and the interceptor places his intercepting unit there.

$p(i, j)$ = probability that the evader will be stopped at arc (i, j) given that arc (i, j) is on his chosen path and the interceptor places his intercepting unit there.

$\pi(i)$ = probability that the interceptor places his intercepting unit at node i.

$\pi(i,j)$ = probability that the interceptor places his intercepting unit at arc (i,j).

The probability that the evader can successfully cross a particular node, i, is $1-\pi(i)p(i)$ and that of crossing a particular arc (i,j) is $1-\pi(i,j)p(i,j)$. Assuming these probabilities are independent* and the evader attempts to reach the sink by the path $i_i, \ldots i_n$, his probability of successful traverse is

$$K=\prod_{r=1}^{n}[1-\pi(i_r)\,p(i_r)]\prod_{r=1}^{n-1}[1-\pi(i_r, i_{r+1})\,p(i_r, i_{r+1})]. \tag{1}$$

This quantity, K, will be referred to as the value of the path $i_1, \ldots, i_n$.

Given the $p(i)$, $p(i, j)$, $\pi(i)$, and $\pi(i, j)$, the evader's problem is to find a source-sink path of maximum value. The interceptor's problem, given the $p(i)$ and $p(i, j)$, is to choose $\pi(i) \geq 0$ and

*This assumption is in fact not true. To illustrate this, suppose $\pi(i) = 1/2$ and $\pi(j) = 1/2$. Independence would require that the probability of the intercepting unit being at nodes i and j at the same time be 1/4 while in reality it is zero. However, the next section will show that this assumption leads to no inaccuracies for the one-intercepting-unit case.

$\pi(i, j) \geq 0$ such that the maximum value of all source-sink paths is minimized, subject to the constraint $\Sigma\pi(i) + \Sigma\pi(i, j) \leq 1$. Note that if all $p(i)$ and $p(i, j)$ are strictly positive, any optimal strategy will be such that strict equality holds in this relationship. To see this, suppose the intercepting unit is not placed at any arc or node with probability p (i.e., $\Sigma\pi(i) + \Sigma\pi(i, j) = 1 - p$). Then increasing all $\pi(i)$ and $\pi(i, j)$ by equal amounts totaling p yields an improved solution.

III. A STEEPEST ASCENT APPROACH FOR ONE INTERCEPTING UNIT

The problem of finding the $\pi(i)$ and $\pi(i, j)$ can in theory be solved by linear programming. The linear program solution would include an optimal strategy for the evader as well as the interceptor. The value of the game would be equal to both the evader's maximum guaranteed probability of successful traverse, and one minus the interceptor's guaranteed probability of stopping the infiltrator. However, the game matrix for the program would require a column for every possible source-sink path and a row for each of the interceptor's pure strategies. The number of source-sink paths is exceedingly large even for relatively small networks and in later sections, when multiple intercepting units are allowed, the same may be said of the number of pure interceptor strategies.

The algorithm of this section is a steepest ascent approach which avoids these enumeration problems; however, while it yields an optimal strategy for the interceptor and a value for the game, it does not yield an optimal strategy for the evader.

Initially, all $\pi(i)$ and $\pi(i, j)$ are assigned values of zero. Then they are increased by small amounts $\Delta\pi(i) \geq 0$ and $\Delta\pi(i, j) \geq 0$. The proportions for the $\Delta\pi(i)$ and $\Delta\pi(i, j)$ are such that the additional intercepting unit placement probability, $\Sigma\Delta\pi(i) + \Sigma\Delta\pi(i, j)$, divided by the decrease in maximum path value, is minimized as these two quantities tend toward zero. This is equivalent to maximizing the decrease in maximum path value per unit of additional intercepting unit probability allocation. The $\pi(i)$ and $\pi(i, j)$ are then increased again in the same manner until they sum up to one.

For a specific path, $i_1, \ldots, i_n$, increasing $\pi(i_r)$ by $\Delta\pi(i_r)$ replaces the factor $[1 - \pi(i_r)p(i_r)]$ in Eq. (1) by $[1 - (\pi(i_r) + \Delta\pi(i_r))p(i_r)]$ and reduces the value of the path from K to $K - \Delta K$, where

$$K - \Delta K = K\left\{\frac{1 - [\pi(i_r) + \Delta\pi(i_r)]x[p(i_r)]}{1 - \pi(i_r)p(i_r)}\right\}. \tag{2}$$

Solving for the quotient of the additional intercepting unit probability allocation and the decrease in path value, the following expression is obtained:

$$\frac{\Delta\pi(i_r)}{\Delta K} = \frac{1}{K}\left[\frac{1}{p(i_r)} - \pi(i_r)\right]. \tag{3a}$$

If intercepting unit placement probability were increased at an arc instead of a node, the same argument would yield an expression for the additional intercepting unit probability placement divided by the decrease in path value which is identical to the preceding one, with arc probabilities substituted for node probabilities. Specifically, for arc (i_r, i_{r+1}), this would be

$$\frac{\Delta\pi(i_r, i_{r+1})}{\Delta K} = \frac{1}{K}\left[\frac{1}{p(i_r, i_{r+1})} - \pi(i_r, i_{r+1})\right]. \tag{3b}$$

Note that in order to obtain any decrease in maximum path value, the $\Delta\pi(i)$ and $\Delta\pi(i, j)$ must be strictly greater than zero along a subset, C, of arcs and nodes that interesects all paths of maximum

value. An efficient allocation would require that all members of C remain on maximum value paths, for otherwise, lessening the probability increase at an arc or node not on a maximum value path by a small amount would decrease $\Sigma\Delta\pi(i)+\Sigma\Delta\pi(i,j)$, while not affecting the maximum path value. Then, of course, this small amount of probability could be redistributed among arcs and nodes on maximum value paths to get a strict improvement. Thus, the $\Delta\pi(i)$ and $\Delta\pi(i,j)$ assigned to the chosen C must be such that the ΔK's are equal for each of its members. In other words, they must be proportional to

(4)
$$\text{a)}\ \frac{1}{p(i)}-\pi(i)$$

$$\text{b)}\ \frac{1}{p(i,j)}-\pi(i,j).$$

The constant $\frac{1}{K}$, where K is the maximum path value, was dropped in Expression (4), of course. The total increase in intercepting unit placement probability per unit decrease in the value of a maximum probability path would be:

(5)
$$\frac{1}{K}\sum_{i\epsilon C}\left[\frac{1}{p(i)}-\pi(i)\right]+\frac{1}{K}\sum_{(i,j)\epsilon C}\left[\frac{1}{p(i,j)}-\pi(i,j)\right].$$

Thus, the problem is reduced to finding a subset of arcs and nodes intersecting all maximum probability paths that minimizes Expression (5).

Initially, all paths have value one since all $\pi(i)$ and $\pi(i,j)$ are zero. Thus C is required to be a set blocking all source-sink paths. If the nodes and arcs are assigned capacities equal to the quantities of Expression (4a) and (4b), the problem of finding C is reduced to one of finding a minimum cut.* However, this is equal to the value of the maximum source-sink flow and can be found by the maximum flow algorithm of Ref. [3].

Note that once additional intercepting unit probability is allocated to the minimum cut, C, the capacity of each arc and node in C decreases by its additional probability increase while the capacity of any other arc or node remains the same. Thus, the value of C will be reduced by $\Sigma\Delta\pi(i)+\Sigma\Delta\pi(i,j)$ while all other cuts will decrease by amounts that do not exceed this. Hence, C remains minimum and the entire unit of probability may be allocated to C.

The $\pi(i)$ and $\pi(i,j)$ may therefore be solved for as follows:

1. Assign all nodes capacities of $1/p(i)$ and all arcs capacities of $1/p(i,j)$ (since all $\pi(i)$ and $\pi(i,j)$ are zero).

2. Maximize flow from source to sink.

3. Let C be the minimum cut set and V its value. Set $\pi(i)=p(i)/V$ for $i\epsilon C$ and $\pi(i,j)=p(i,j)/V$ for $(i,j)\epsilon C$. Set all other $\pi(i)$ and $\pi(i,j)$ equal to zero.

Note that since only one cut is obtained, each path of maximum value has positive intercepting unit probability allocated to only one of its arcs or nodes, and both its value and traverse probability are equal to the probability associated with that particular arc or node. Hence, the independence assumption does not lead to inaccuracies for the one-intercepting-unit case. Finally, the maximum path value is $1-\frac{1}{V}$, where V is the value of C found in step 3.

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*Essentially, a cut set is a set of arcs and nodes blocking all source-sink paths. Its value is the sum of the capacities of its arcs and nodes, and the minimum value of all cuts is equal to the maximum value of the source-sink flow. See Ref. [3].

IV. MULTIPLE INTERCEPTING UNITS

The steepest ascent algorithm of the last section may be modified to handle the case where there is more than one intercepting unit; however, the interceptor strategy obtained in some cases may not be optimal and the game value obtained may be approximate rather than exact. This will be discussed more fully later on. The quantities measuring the local effectiveness of placing intercepting units at arcs and nodes and those defining the interceptor's strategy must be redefined as follows:

$p(i)_k$ = probability that the evader will be stopped at node i, given that node i is on his chosen path and the interceptor places k intercepting units there.

$p(i,j)_k$ = probability that the evader will be stopped at arc (i,j), given that arc (i,j) is on his chosen path and the interceptor places k intercepting units there.

$\pi(i)$ = expected number of intercepting units the interceptor places at node i.

$\pi(i,j)$ = expected number of intercepting units the interceptor places at arc (i,j),

where $p(i)_0$ and $p(i,j)_0$ are both identically zero. Also let

$$\tilde{\pi}(i) = \pi(i) - [\pi(i)]$$
$$\tilde{\pi}(i,j) = \pi(i,j) - [\pi(i,j)].$$

Note that $\tilde{\pi}(i)$ and $\tilde{\pi}(i,j)$ are merely the fractional parts of $\pi(i)$ and $\pi(i,j)$. It will be assumed $p(i)_k$ and $p(i,j)_k$ are increasing in k and

(6) a) $p(i)_{k+1} - p(i)_k \leq p(i)_k - p(i)_{k-1}$

b) $p(i,j)_{k+1} - p(i,j)_k \leq p(i,j)_k - p(i,j)_{k-1}$, $k > 1$.

The assumption of Expression (6) essentially states that if an additional intercepting unit is placed at any arc or node, it cannot be more effective in stopping the evader than any previous unit placed there. It is often satisfied in practice as the effects of overkill become more pronounced as the number of units at a location increase. The simplest example of this is the case where each unit placed at node i has independent probability p of stopping the evader. For this situation $p(i)_k = 1-(1-p)^k$ and $p(i)_{k+1} - p(i)_k = p(1-p)^k$ which is certainly decreasing in k, thus satisfy (6).

The values of $\pi(i)$ and $\pi(i,j)$ of course are not sufficient in themselves to define the interceptor's strategy as any non-zero value of $\pi(i)$ or $\pi(i,j)$ may often be realized in several ways. However, the assumption of Expression (6) assures that for a given value of $\pi(i)$, the probability of stopping the evader at node i should he attempt to cross it is minimized by allocating $[\pi(i)]+1$ units at i with probability $\tilde{\pi}(i)$, and $[\pi(i)]$ units with probability $1-\tilde{\pi}(i)$. In other words, if $\pi(i) = 3\text{-}1/2$, $\pi(i)$ would best be realized by assigning three forces at i half the time and four half the time, as opposed to such a policy as allocating two forces half the time and five half the time. This is shown in the theorem to follow. First, define $P\{k, i\}$ as the probability that k units are placed at node i. Note that $\Sigma k P\{k, i\} = \pi(i)$ and the probability of thwarting an evader's attempt to cross node i is $1 - \Sigma P\{k, i\} p(i)_k$. Minimizing this probability is equivalent to maximizing $\Sigma P\{k, i\} p(i)_k$.

THEOREM: The quantity $(i)\Sigma P\{k, i\} p(i)_k$ is maximized, subject to the constraint $(ii)\Sigma k P\{k, i\} = \pi(i)$ when $P\{[\pi(i)]+1, i\} = \tilde{\pi}(i)$, $P\{[\pi(i)], i\} = 1-\tilde{\pi}(i)$, and all other $P\{k, i\} = 0$.

PROOF: Let $h = \max\{k/P\{k, i\} > 0\}$, $l = \min\{k/P\{k, i\} > 0\}$, and $d = h - l$. Let $\bar{P}\{k, i\} = P\{k, i\}$ be a set of values of smallest d which maximizes (i) subject to (ii). Suppose the theorem is false. Then $h - l = d \geq 2$. Let $m = \min\{\bar{P}\{h, i\}, \bar{P}\{l, i\}\}$. Set $P\{h, i\} = \bar{P}\{h, i\} - m$ and $P\{h-1, i\} = \bar{P}\{h-1, i\} + m$. Then set $P\{l, i\} = \bar{P}\{l, i\} - m$ and increase $P\{l+1, i\}$ by m. (If $h - l = 2$, then $P\{l+1\} = \bar{P}\{l+1\} + 2m$.) Either $P\{h, i\}$ or $P\{l, i\}$ is now zero, decreasing d by at least one unit.

The increase in (i) is equal to $m(-p(i)_l+p(i)_{l+1}-p(i)_h+p(i)_{h-1})$. Since (ii) is still satisfied, this is strictly negative from our choice of $\bar{P}\{k, i\}$. However, since $h>l+1$,

$$p(i)_h-p(i)_{h-1}\leqslant p(i)_{l+1}-p(i)_l$$

$$-p(i)_l+p(i)_{l+1}-p(i)_h+p(i)_{h-1}\geqslant 0$$

for a contradiction. Q.E.D.

The same results, of course, hold for the $\pi(i, j)$ of the arcs. The significance of this is that the search for an optimal interceptor strategy may be confined to strategies of the above type. When this is done, the $\pi(i)$ and $\pi(i, j)$ are sufficient to specify the interceptor's strategy.

As in Eq. (1) of Sec. II, if independence is assumed, the value of a path is still the product of the probabilities associated with its nodes and arcs; however, the probability associated with node i is now

$$\tilde{\pi}(i)\,(1-p(i)_{[\pi(i)]+1})+(1-\tilde{\pi}(i))\,(1-p(i)_{[\pi(i)]})$$

or

$$1-p(i)_{[\pi(i)]}-\tilde{\pi}(i)\,(p(i)_{[\pi(i)]+1}-p(i)_{[\pi(i)]}),$$

and that with arc (i, j) is

$$1-p(i, j)_{[\pi(i, j)]}-\tilde{\pi}(i, j)\,(p(i, j)_{[\pi(i, j)]+1}-p(i, j)_{[\pi(i, j)]}).$$

If $i_1, \ldots, i_n$ is a path of value K, and $\pi(i_r)$ is increased by $\Delta\pi(i_r)$, reducing the path value to $K-\Delta K$, the expressions for K and $K-\Delta K$, as before, differ only in the factor for node i_r. Thus, for sufficiently small $\Delta\pi(i_r)$, the expression for the quotient of the intercepting unit increase and path decrease is

$$\frac{\Delta\pi(i_r)}{\Delta K}=\frac{1}{K}\left\{\frac{1-p(i_r)_{[\pi(i_r)]}}{p(i_r)_{[\pi(i_r)]+1}-p(i_r)_{[\pi(i_r)]}}-\tilde{\pi}(i_r)\right\},$$

and decreasing path value by increasing the number of intercepting units at an arc instead of a node yields the analogous expression,

$$\frac{\Delta\pi(i_r, i_{r+1})}{\Delta K}=\frac{1}{K}\left\{\frac{1-p(i_r, i_{r+1})_{[\pi(i_r, i_{r+1})]}}{p(i_r, i_{r+1})_{[\pi(i_r, i_{r+1})]+1}-p(i_r, i_{r+1})_{[\pi(i_r, i_{r+1})]}}-\pi(i_r, i_{r+1})\right\}.$$

As in Sec. III, intercepting units must be allocated along all arcs of some cut set, and the problem is reduced to finding a minimum cut set, which in turn reduces to one of finding a maximum source-sink flow. The node and arc capacities are now

(7) a)
$$c(i)=\frac{1-p(i)_{[\pi(i)]}}{p(i)_{[\pi(i)]+1}-p(i)_{[\pi(i)]}}-\tilde{\pi}\ (i)$$

b)
$$c(i, j)=\frac{1-p(i, j)_{[\pi(i, j)]}}{p(i, j)_{[\pi(i, j)]+1}-p(i, j)_{[\pi(i, j)]}}-\tilde{\pi}\ (i, j).$$

Note that the $c(i)$ and $c(i, j)$ are no longer decreasing functions of $\pi(i)$ and $\pi(i, j)$ except over intervals whose endpoints possess the same integer part. Specifically, $c(i)$ is a decreasing function of

$\pi(i)$ for $0 \leq \pi(i) < 1$, $1 \leq \pi(i) < 2$, etc. However, $c(i)$ is discontinuous and actually increases at all integer points for which (6a) holds with strict inequality. This follows since

$$\lim_{\tilde{\pi}(i) \to 1} \frac{1-p(i)_{[\pi(i)]}}{p(i)_{[\pi(i)]+1}-p(i)_{[\pi(i)]}} - \tilde{\pi}(i) = \frac{1-p(i)_{[\pi(i)]+1}}{p(i)_{[\pi(i)]+1}-p(i)_{[\pi(i)]}} < \frac{1-p(i)_{[\pi(i)]+1}}{p(i)_{[\pi(i)]+2}-p(i)_{[\pi(i)]+1}}.$$

Similar results hold for $c(i, j)$. Thus, when a minimum cut set is found and additional intercepting units are allocated along its arcs and nodes, that cut remains minimum provided the expected number of intercepting units on these arcs and nodes do not reach or exceed their next highest integer values. This requires that the steepest ascent approach be modified as follows.

Starting with all $\pi(i)$ *and* $\pi(i, j)$ equal to zero, capacities are assigned to the arcs and nodes of the network which are equal to the quantities defined by Expression (7). Flow is maximized from source to sink and the $\pi(i)$ and $\pi(i, j)$ corresponding to the arcs and nodes of the minimum cut set are increased by amounts $\Delta\pi(i)$ and $\Delta\pi(i, j)$ which are proportional to their capacities. The constant of proportionality, M, is the smallest possible such constant that will either increase some $\pi(i)$ or $\pi(i, j)$ to the next highest integer or will increase $\Sigma\pi(i) + \Sigma\pi(i, j)$ to n. If the latter does not happen, the new values of $\pi(i)$ and $\pi(i, j)$ are used to calculate new capacities and flow is maximized again to obtain a new cut. The process is repeated until $\Sigma\pi(i) + \Sigma\pi(i, j) = n$.

Specifically, the algorithm for the general case of n intercepting units is as follows:

1. Set all $\pi(i) = 0$ and all $\pi(i, j) = 0$.
2. Assign the nodes and arcs of the network capacities of $c(i)$ and $c(i, j)$ respectively, where $c(i)$ and $c(i, j)$ are as defined by Expression (7).
3. Maximize flow from source to sink and let C be the nodes and arcs of the minimum cut set found.
4. Compute $M_1 = \min_{i \in C} \dfrac{\{1-\tilde{\pi}(i)\}}{c(i)}$

$$M_2 = \min_{(i,j) \in C} \frac{\{1-\tilde{\pi}(i, j)\}}{c(i, j)}$$

$$M_3 = \frac{n - \sum \pi(i) - \sum \pi(i, j)}{\sum_{i \in C} c(i) + \sum_{(i,j) \in C} c(\mathrm{i}, \mathrm{j})}.$$

5. Let $M = \min\ (M_1, M_2, M_3)$. Set $\Delta\pi(i) = Mc(i)$ if $i \in C$ and $\Delta\pi(i, j) = Mc(i, j)$ if $(i, j) \in C$. Set all other $\Delta\pi(i)$ and $\Delta\pi(i, j)$ equal to zero.
6. Set $\pi(i) = \pi(i) + \Delta\pi(i)$ and $\pi(i, j) = \pi(i, j) + \Delta\pi(i, j)$ for all i and (i, j). If $\Sigma\pi(i) = \Sigma\pi(i, j) < n$, go back to step 2. Otherwise terminate.

Step 1 is simply an initialization step. Steps 2 and 3 determine which nodes and arcs should have their expected number of intercepting units increased and the proportions of the increase. Specifically the expected number of intercepting units will be increased along the nodes and arcs of the minimum cut set found in proportion to their capacities. Steps 4 through 6 determine the proportionality constant M, makes the appropriate increases, and makes a termination test. Note that if $M = M_1$, some $\pi(i)$ will be increased by $1 - \tilde{\pi}(i)$, and since $\tilde{\pi}(i)$ is the fractional part of $\pi(i)$, this will increase $\pi(i)$ to an integer value. Similarly if $M = M_2$, some $\pi(i, j)$ will be increased to an integer value. If $M = M_3$, total expected intercepting units will be increased by $n - \Sigma\pi(i) - \Sigma\pi(i, j)$ to n. In the latter case all n intercepting units have been allocated and the procedure terminates.

At termination the $\pi(i)$ and $\pi(i, j)$ represent the expected number of intercepting units to place at node i and arc (i, j), respectively. As mentioned earlier, the strategy will be to place $[\pi(i)]+1$ forces at node i with probability $\tilde{\pi}(i)$ and $[\pi(i)]$ forces with probability $1-\tilde{\pi}(i)$.

The maximum path value can be kept track of during the course of the algorithm. Specifically, if K is the maximum path value at the beginning of an iteration, then the maximum path value at the end of the iteration is $K(1-M)$, where M is that found in Step 5. Of course, $K=1$ at the beginning of the first iteration. This is easily proven by letting V be the value of C, the minimum cut set. Then substitute the generalized capacities (Expression 7a–b) for the specialized ones (Expression 4a–b) in Expressions (3) and (5). These replacements for Expressions (3) and (5) will have the same meaning in the multiple intercepting unit situation as Expressions (3) and (5) did for the single intercepting unit situation. The replacement for (5), which represents the total increase in intercepting unit placement probability per unit decrease in the value of a maximum probability path, is:

$$\frac{\Sigma\Delta\pi(i)+\Sigma\Delta\pi(i, j)}{\Delta K}=\frac{1}{K}V.$$

However, $\Sigma\Delta\pi(i)+\Sigma\Delta\pi(i, j)=MV$ where M is as defined in Step 5 of the multiple intercepting unit algorithm. Thus

$$\frac{MV}{\Delta K}=\frac{1}{K}V$$

or

$$K-\Delta K=K(1-M).$$

Note that when there is more than one intercepting unit to be assigned, the strategy obtained is not necessarily optimal. To illustrate this, consider the network of Fig. 1.

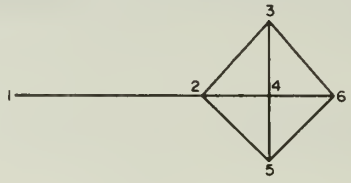

FIGURE 1. A simple network with node 1 the source and node 6 the sink.

Suppose $p(1, 2)_1=5/8$, $p(1, 2)_2=1$, $p(2)_k=1-(1/4)^k$, and all other $p(i)_k$ and $p(i, j)_k$ equal $1-(1/2)^k$. If there were two intercepting units available, placing both at arc (1, 2) with probability one will make the evaders successful probability of traverse equal to zero; however, the algorithm, as may be easily verified, will place both units at node 2 with probability one giving the evader a successful traverse probability of 1/16. For one unit, however, the algorithm will optimally place it at node 2 giving the evader a successful traverse probability of 1/4.

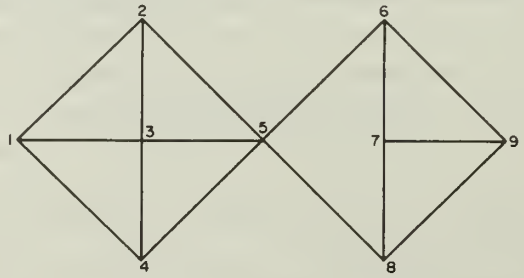

FIGURE 2. Sample problem network with node 1 as source and node 9 as sink.

V. EXAMPLE

For illustrative purposes, the algorithm is used to find the optimal placement of forces for the network of Fig. 2, with the results of each iteration. Ten intercepting units are available, and all arcs and nodes have identical probabilities associated with them. Thus, for each node, i, and each arc (i, j), we have:

$$p(i)_1 = 0.6666 \qquad p(i,j)_1 = 0.6666$$
$$p(i)_2 = 0.8332 \qquad p(i,j)_2 = 0.8332$$
$$p(i)_3 = 0.8748 \qquad p(i,j)_3 = 0.8748$$
$$p(i)_4 = 0.8852 \qquad p(i,j)_4 = 0.8852$$
$$p(i)_5 = 0.8878 \qquad p(i,j)_5 = 0.8878$$
$$p(i)_6 = 0.8884 \qquad p(i,j)_6 = 0.8884$$
$$p(i)_7 = 0.8886 \qquad p(i,j)_7 = 0.8886$$
$$p(i)_8 = 0.8887 \qquad p(i,j)_8 = 0.8887$$
$$p(i)_9 = 0.8888 \qquad p(i,j)_9 = 0.8888$$
$$p(i)_{10} = 0.8889 \qquad p(i,j)_{10} = 0.8889$$

The problem terminates in eight iterations. The results of these iterations follow:

Iteration 1: All arcs and nodes have capacities of 1.5. A maximum flow is 1 unit along path 1,2,5,6,9. A minimum cut set is node 1 and its value is 1.5. Increasing intercepting units at node 1 to 1 gives a maximum path value of 0.33333.

Iteration 2: The capacity of node 1 is increased to 2.0; all other capacities are unchanged from iteration 1. A maximal flow is 1.5 units along path 1,2,5,6,9. A minimum cut set is node 5 and its value is 1.5. Increasing intercepting units at node 5 to 1 gives a maximum path value of 0.11111.

Iteration 3: The capacity of node 5 is increased to 2.0; all others are unchanged. A maximal flow is 1.5 units along path 1,2,5,6,9. A minimum cut set is node 9 and its value is 1.5. Increasing intercepting units at node 9 to 1 gives a maximum path value of 0.03704.

Iteration 4: The capacity of node 9 is increased to 2.0; all others are unchanged. A maximal flow is 1.5 units along path 1,2,5,6,9 and 0.5 unit along path 1,3,5,8,7,9. A minimum cut set is node 1 and its value is 2.0. Increasing intercepting units at node 1 to 2 gives a maximum path value of 0.01853.

Iteration 5: The capacity of node 1 is increased to 4.0; all others are unchanged. A maximum flow is 1.5 units along path 1,2,5,6,9 and 0.5 unit along path 1,3,5,8,7,9. A minimum cut set is node 5 and its value is 2.0. Increasing intercepting units at node 5 to 2 gives a maximum path value of 0.00927.

Iteration 6: The capacity of node 5 is increased to 4.0; all others are unchanged. A maximal flow is 1.5 units along path 1,2,5,6,9 and 0.5 unit on path 1,3,5,8,7,9. A minimum cut set is node 9 and its value is 2.0. Increasing intercepting units at node 9 to 2 gives a maximum path value of 0.00464.

Iteration 7: The capacity of node 9 increased to 4.0; all others are unchanged. A maximal flow is 1.5 units along path 1,2,5,6,9 and 1.5 units along path 1,3,5,8,7,9. A minimum cut set is arcs (5,6) and (5,8) and its value is 3.0. Increasing intercepting units to 1 at arcs (5,6) and (5,8) gives a maximum path value of 0.00155.

Iteration 8: The capacities of arcs (5,6) and (5,8) are increased to 2.0; all others are unchanged. A maximal flow is 1.5 units along path 1,2,5,6,9 and 1.5 units along path 1,3,5,8,7,9. A minimum cut set is nodes 6 and 8 and its value is 3.0. Increasing intercepting units to 1 at nodes 6 and 8 gives a maximum path value of 0.00052. Algorithm terminates as all 10 units have been assigned.

The final assignment is 2.0 intercepting units each at nodes 1,5, and 9; and 1.0 intercepting unit each at nodes 6 and 8 and arcs (5,6) and (5,8).

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THE FIXED CHARGE PROBLEM*

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ABSTRACT

The fixed charge problem is a nonlinear programming problem of practical interest in business and industry. Yet, until now no computationally feasible exact method of solution for large problems had been developed.

In this paper an exact algorithm is presented which is computationally feasible for large problems. The algorithm is based upon a branch and bound approach, with the additional feature that the amount of computer storage required remains constant throughout (for a problem of any given size).

Also presented are three suboptimal heuristic algorithms which are of interest because, although they do not guarantee that the true optimal solution will be found, they usually yield very good solutions and are extremely rapid techniques.

Computational results are described for several of the heuristic methods and for the branch and bound algorithm.

1. INTRODUCTION

The fixed charge problem is perhaps one of the most intriguing problems of mathematical programming. Its structure is almost identical with that of a linear programming problem; yet, the one difference—the existence of the fixed charges in its objective function—has prevented the development of any extensive theory concerning its solution.

There are two known exact methods of solution: total enumeration of the extreme points of the convex set of feasible solutions, and a mixed integer-continuous variable formulation. However, total enumeration is impractical for large problems, and, at this time, methods for solving mixed integer-continuous variable problems involving a large number of integer variables are not effective enough to be of much use, either. Therefore, heuristic methods for obtaining a "good" (although not necessarily optimal) solution have been developed. Some of these methods will be discussed in Section 2.

Also presented is an exact algorithm, based on a branch and bound approach, which has been developed by the author, along with several new heuristic methods. Computational results are given in Section 4.

1.1 Formulation of the Fixed Charge Problem

The fixed charge problem is a nonlinear programming problem of the form:

$$\text{Minimize} \quad z = cx + dy \tag{1}$$

$$\text{subject to:} \quad Ax = b, \quad x \geq 0, \tag{2}$$

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where A is an $m \times n$ matrix of real numbers,

b is an m-vector in an m-dimensional space, E^m,

c and d are n-vectors in E^n,

x is an n-vector of unknowns, and

y is an n-vector such that

$$y_j = \begin{cases} 0, & \text{if } x_j = 0 \\ 1, & \text{if } x_j > 0 \end{cases} \tag{3}$$

It is assumed that all $d_j \geq 0$. The quantities d_j are the fixed charges which are incurred if the corresponding x_j are (strictly) positive. If all $d_j = 0$, then (1) and (2) reduce to a linear programming problem.

It is known [7] that the optimal solution to the fixed charge problem occurs at an extreme point of the convex set of feasible solutions (2). Since the total number of extreme points is finite, the problem could be solved by evaluating the objective function (1) at every extreme point and taking the minimum of these quantities. However, as is the case with linear programming, the number of extreme points increases combinatorially with m and n. Hence, it is desirable to develop a method that requires that only a subset of the extreme points be examined in order to determine the optimal solution. Two such methods are the mixed integer-continuous variable integer programming formulation and the author's branch and bound algorithm.

1.2 A Mixed Integer-Continuous Variable Approach

Hadley [7] presents the following formulation of the fixed charge problem and suggests that it be solved by use of Gomory's mixed integer-continuous variable algorithm.

$$\text{Minimize } z = cx + dy,$$

$$\text{subject to: } Ax = b$$

$$x_j - U_j y_j \leq 0, \qquad j = 1, \ldots, n$$

$$0 \leq y_j \leq 1, \qquad j = 1, \ldots, n$$

$$y_j \text{ integers}$$

$$x \geq 0,$$

where the U_j represent known upper bounds for the x_j.

If A is $m \times n$, then the above formulation has $m + 2n$ constraints. It should be noted, however, that the upper bound constraints on the y_j need not be expressed explicitly, since no optimal solution would have $y_j > 1$. Thus, the actual basis size required to use Gomory's algorithm is $m + n$. However, computational experience with Gomory's algorithm has been very disappointing, even for some fairly small problems [7]; therefore, this approach is not widely used to solve fixed charge problems.

2. SOME HEURISTIC METHODS

The fact that the optimal solution to the fixed charge problem lies at an extreme point of the convex set (2) has led to the development of some heuristic approaches which use adjacent extreme point techniques to obtain good solutions. All of the methods discussed below are basically adjacent extreme point methods.

There are also several heuristic methods for solving the fixed charge transportation problem. These are presented in [2] and [3], but shall not be discussed here.

The following notations are offered prior to presenting the heuristic methods:

x_B = vector of basis variables,
c_B = vector of prices for variables in the basis,
a_j = jth column of A,
B = basis matrix,
$y_j = B^{-1}a_j$, and
$z_j = c_B y_j$.

From the theory of linear programming [6, 7] it is known that once a vector a_j is chosen to enter the basis, the vector a_{Br} which leaves the basis is determined by

$$\frac{x_{Br}}{y_{rj}} = \operatorname*{Min}_k \{x_{Bk}/y_{kj}, \text{ for } y_{kj} > 0\}. \tag{4}$$

If all $d_k = 0$ (i.e., the problem is all linear) the greatest decrease in the objective function at each iteration is obtained if the vector to enter the basis, a_j, is chosen so that

$$\frac{x_{Br}}{y_{rj}}(c_j - z_j) = \operatorname*{Min}_k \left\{ \frac{x_{Br}}{y_{rk}}(c_k - z_k), \qquad \text{for } y_{rk} > 0 \right\}. \tag{5}$$

When this expression is nonnegative the optimal solution has been reached. For the fixed charge problem (one or more $d_j \neq 0$) criterion (5) must be modified to account for the fixed charges; the greatest decrease in z is obtained if a_j is chosen so that

$$e_j = \operatorname*{Min}_k \{e_k, \text{ for } y_{rk} > 0\}, \tag{6}$$

where

$$e_k = \frac{x_{Br}}{y_{rk}}(c_k - z_k) + (d_{Br} - d_k) \tag{7}$$

Equation (7) must be modified slightly in the presence of degeneracy, in an obvious way. (It should be noted that the variable which leaves the basis, x_{Br}, will, in general, be different for each j). Because the objective function is concave, it is not true that when all $e_k \geq 0$ a global minimum has been reached. This stage may merely be a local minimum. That is, no adjacent extreme point yields an improved value of z, but it is still possible that some other extreme point of the convex set will yield a better value of z. The methods discussed below differ in their approach to the problem of trying to obtain a better solution once a local minimum has been reached. The first three methods are those of the author.

2.1 Heuristic One

1. By using a Simplex Algorithm, vectors are removed from and entered into the basis according to (4) and (6), respectively, until all $e_j \geq 0$. Let the current solution be denoted by x_0, with corresponding value of the objective function z_0. Store x_0. Define two parameters:

α_0 = maximum number of consecutive iterations allowed to find a new extreme point with a value of z smaller than z_0.

β_0 = maximum number of consecutive iterations allowed in which the objective function does not decrease.

Also, define two "counters," α_1 and β_1, which indicate the current number of consecutive iterations for which z_0 has not been improved, and in which the objective function does not decrease, respectively. Set $\alpha_1 = \beta_1 = 0$.

2. Find the e_j which yields the smallest increase in z and insert a_j into the basis. Set $\alpha_1 = \beta_1 = 1$. Proceed to Step 3.

3. Find the e_j which yields the largest decrease in z and insert a_j into the basis. Proceed to Step 4.

4. Compare the new z, z_1, with z_0:

a) If $z_1 < z_0$, return to Step 1.

b) If $z_1 = z_0$, but $x_1 \neq x_0$, set $\alpha_1 = \alpha_1 + 1$. If $\alpha_1 \leq \alpha_0$ return to Step 3. Otherwise terminate.

c) If $z_1 = z_0$ and $x_1 = x_0$, set $\alpha_1 = \alpha_1 + 1$ and $\beta_1 = \beta_1 + 1$. If $\alpha_1 \leq \alpha_0$ and $\beta_1 \leq \beta_0$ go to Step 5. Otherwise terminate.

d) If $z_1 > z_0$, set $\alpha_1 = \alpha_1 + 1$. If $\alpha_1 \leq \alpha_0$ return to Step 3. Otherwise terminate.

5. Perform β_1 consecutive interations in each of which the vector which yields the largest increase in z is inserted into the basis. Return to Step 3.

This algorithm attempts to find better solutions by moving away from the current local minimum in as few iterations as possible. The first such attempt (Step 2) actually increases the objective function as little as possible; if this attempt fails, the objective function is increased as much as possible (Step 5) with the hope that upon once again decreasing (Step 3) a better local minimum will be found. The efficiency of this method, as well as its accuracy, depends on the parameters α_0 and β_0. In general, α_0 should probably be somewhat larger than β_0. Since computational experience [4] has shown that the average number of iterations required to solve a linear programming problem with m constraints is about $3/2m$, a reasonable range for α_0 might be $\alpha_0 \epsilon [m/2, m]$. As the computational results of Section 4 indicate, much smaller values may also be used fairly successfully. Of course, economic considerations may also play a part in the determination of α_0 and β_0.

2.2 Heuristic Two

1. Same as in previous heuristic, except that the entire Simplex tableau corresponding to x_0 is saved. Define α_0 as before, set $\beta_0 = n - m$, $\alpha_1 = 1$, $\beta_1 = 1$, and proceed to Step 2 (assuming again that all $e_j \geq 0$ at x_0).

2. Beginning with the tableau corresponding to x_0, find the e_j which yields the (β_1)th smallest increase in z and insert a_j into the basis. Set $\beta_1 = \beta_1 + 1$, and if $\beta_1 \leq (\beta_0 + 1)$ proceed to Step 3. Otherwise terminate.

3. Find the e_j which yields the largest decrease in z and insert a_j into the basis. Set $\alpha_1 = \alpha_1 + 1$ and if $\alpha_1 \leq \alpha_0$, proceed to Step 4. If $\alpha_1 > \alpha_0$, proceed to Step 2.

4. Compare z_1 with z_0:

a) If $z_1 > z_0$, return to Step 3.

b) If $z_1 = z_0$, but $x_1 \neq x_0$, return to Step 3.

c) If $z_1 = z_0$ and $x_1 = x_0$, set $\alpha_1 = 1$ and return to Step 2.

d) If $z_1 < z_0$, return to Step 1.

2.3 Heuristic Three

Heuristic Three employs a Monte Carlo approach to find an extreme point with a better value of z than x_0 (where again at x_0 all $e_j \geq 0$). In this method, extreme points within a certain "distance" from x_0 are randomly examined. "Distance" here shall mean the number of basis changes required to move from one extreme point to another. Let $\delta =$ maximum distance from x_0 allowed and let α_0 again be the maximum number of iterations allowed to find an improvement in z.

The procedure is as follows:

1. Insert a_j into the basis, where e_j is the maximum decrease in z. Continue in this manner until

all $e_j \geq 0$. Denote the current extreme point by x_0. Set $\alpha_1 = 0$ and proceed to Step 2.

2. Randomly select (according to the procedure described below) a vector within a distance δ of x_0. Denote this vector by x_1. Set $\alpha_1 = \alpha_1 + 1$ and proceed to Step 3.

3. Compare z_1 with z_0:

a) If $z_1 > z_0$ and $\alpha_1 \geq \alpha_0$, return to Step 2. If $z_1 > z_0$ and $\alpha_1 > \alpha_0$, terminate.

b) If $z_1 \leq z_0$, return to Step 1, replacing x_0 and z_0 by x_1 and z_1, respectively.

In order to complete the description of the above algorithm, the procedure for the random selection of extreme points must be described. For simplicity, the vectors selected to enter the basis are chosen one at a time (and the corresponding vector to leave is determined by (4)). The basis transformation is then performed before the next vector to enter the basis is again chosen randomly. Since there are $(n-m)$ nonbasis vectors, the random selection of a vector to enter the basis is performed by generating a random number from a uniform random distribution on the interval (0, 1), multiplying it by $(n-m)$, and rounding the result to the next highest integer, say k. Then the kth nonbasic vector is selected to enter the basis. The process is repeated δ times, and the resulting extreme point is the first randomly generated extreme point within a distance δ of x_0 to be used in Step 2.

An approach very similar to the above is proposed by D. Denzler [5]; the basic difference is that Denzler's method does not restrict its random search for an extreme point within a certain distance of the current best solution. Many iterations are thus wasted. Intuitively, it would seem that extreme points close to one which has a known good value of z should also yield fairly good values of z, and so it would appear to be a better procedure to restrict the search for better solutions to extreme points fairly close to x_0 rather than attempt to search randomly the entire set of extreme points.

2.4 The Cooper-Drebes and Cooper-Olson Heuristics

Cooper and Drebes [3] have developed two other adjacent extreme point heuristics, which differ from the above three methods mainly in that in the Cooper-Drebes methods the objective function is modified at certain stages in the algorithms; in particular, the linear costs c_j, $j = 1, \ldots, n$, are modified according to

$$\hat{c}_j = c_j + \frac{d_j}{x_j}, \quad x_j > 0 \tag{8}$$

$$\hat{c}_j = c_j, \quad x_j = 0,$$

where "x_j" implies the current value of variable x_j. At other stages in their heuristics, Cooper and Drebes remove from the basis the vector with the largest fixed charge (of the current basic variables) and insert into the basis the vector with the smallest fixed charge (of the current nonbasic variables). The two methods differ only in their decision rules for employing the various techniques discussed above.

A modification of the Cooper-Drebes heuristics has been developed by Cooper and Olson [4]. The modification consists of randomly generating an extreme point a distance not more than δ (where distance is defined as in Section 2.3) from the current local minimum, and then the Cooper-Drebes heuristic is again employed in an attempt to improve the solution. The random extreme point generation is called a "perturbation." The parameter δ is allowed to vary from about $m/3$ to $m/2$. If the search following the last previous perturbation yields a local minimum worse than the best obtained so far, δ is reduced; if the objective function value is the same as the best, δ is increased; if a better value is obtained, δ is not changed.

The computational experience reported on these methods [3, 4] indicates that they will yield the optimal solution a high percentage of the time and, moreover, can be expected to provide at least a near-optimal solution.

2.5 A Separable Programming Approximation Technique

Another technique which yields a local minimum is to formulate the fixed charge problem as a separable programming problem [7].

Letting U_j represent known upper bound values of x_j, consider the following representation of the objective function (1):

$$z=\sum_{j=1}^{n}[d_j\lambda_{1j}+(d_j+c_jU_j)\lambda_{2j}], \tag{9}$$

where

$$x_j=\lambda_{2j}U_j \tag{10}$$

and

$$\lambda_{0j}+\lambda_{1j}+\lambda_{2j}=1 \tag{11}$$

and where no more than two λ_{kj} can be positive for a given j, and these can only be positive if they are adjacent. The complete formulation becomes:

$$\text{Minimize } z=\sum_{j=1}^{n}[d_j\lambda_{1j}+(d_j+c_jU_j)\lambda_{2j}],$$

subject to

$$\sum_{j=1}^{n}a_{ij}U_j\lambda_{2j}=b_i, \qquad i=1, \ldots, m$$

$$\lambda_{0j}+\lambda_{1j}+\lambda_{2j}=1, \qquad j=1, \ldots, n$$

$$\lambda_{kj}\geq 0,\ k=0, 1, 2 \quad \text{and} \quad j=1, \ldots, n.$$

When this problem is solved using a restricted basis entry Simplex method [7], the optimal solution will be a local minimum for the fixed charge problem. There is no evidence, however, that this procedure is any better than solving the fixed charge problem by using linear programming and ignoring the fixed charges. Moreover, no attempts are made to improve the local minimum obtained.

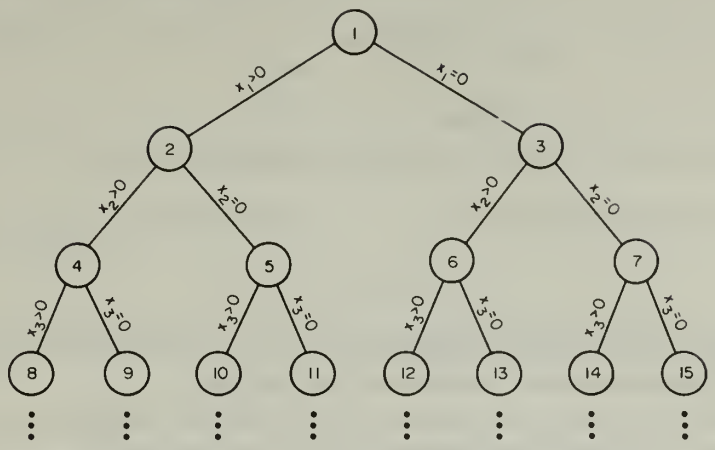
3. A BRANCH AND BOUND ALGORITHM

In attempting to develop an exact solution method for the fixed charge problem, it would be natural to begin by investigating the possibility of developing an adjacent extreme point method. However, since the objective function is not always monotonic (that is, there may not always exist a monotonically decreasing adjacent extreme point path to the optimal solution) some other approach is called for. In this section a branch and bound approach is presented. (A general knowledge of the branch and bound philosophy will be assumed in the following discussion. The reader is referred to [1] for a general discussion of branch and bound methods.)

The algorithm finds the global minimum solution without having to examine all basic feasible solutions of (2). In the next two sections, the general approach will be presented and the computational details will be given in Section 3.3.

3.1 The Tree of Solutions

Suppose the following tree is generated:



The nodes are numbered from left to right and from top to bottom, so that each horizontal "line" of nodes may be thought of as a *level*; that is, the first level consists of only node 1, the second level consists of nodes 2 and 3, the third level consists of nodes 4, 5, 6, and 7, etc. The constraints (2) will be required to be satisfied at every node. In addition, each node (except node 1) will be required to satisfy some additional constraints, of the form "$x_j = 0$" or "$x_j > 0$." Since only basic feasible (extreme point) solutions are of interest, the latter constraints actually require variable x_j to appear in the basic solution, at a positive level. At each node (except node 1) one additional constraint is imposed, so that the solution obtained at any given node k must satisfy this new constraint plus the constraints imposed at every other node in the path from node 1 to node k. Thus, a node at level L must satisfy $(L-1)$ additional constraints.

If a path is terminated when the additional constraints imposed by the node are such that:

1. There are m constraints of the form "$x_j > 0$", or
2. There are $(n-m)$ constraints of the form "$x_j = 0$", or
3. No feasible solution exists when the current additional constraints are imposed,

then it is clear that the collection of terminal nodes will contain all the basic feasible solutions to (2). That is, for terminal case 1, m x_j's are required to be positive and hence a basis is uniquely determined. For case 2, $(n-m)$ x_j's must be zero, and hence the remaining x_j's uniquely determine a basis; however, this basis may not yield a feasible solution, in which case terminal case 3 applies.

Now, of course, the value of the objective function z can be computed for each extreme point and the minimal value obtained; however, it is desirable to reduce as much as possible the number of extreme points which must be examined.

3.2 The Algorithm

The following notations will be introduced:

$$S_1^k = \{j | x_j \text{ must be positive at node } k\},$$

$$S_2^k = \{j | x_j \text{ must be zero at node } k\},$$

$$z^k = \text{a lower bound of } z \text{ at node } k, \text{ and}$$

$$z^* = \text{the global optimal value of } z.$$

The constraint set at node k thus becomes:

$$T^k = \begin{cases} Ax = b \\ x \geq 0 \\ x_j > 0, \quad j \epsilon S_1^k \\ x_j = 0, \quad j \epsilon S_2^k \\ x \text{ is an extreme point.} \end{cases} \tag{13}$$

Suppose that an upper bound estimate of z^*, $\tilde{z}$, is obtained. Then a path can also be terminated at node k if

$$z^k \geq \tilde{z}, \tag{14}$$

since the basic feasible solution corresponding to $\tilde{z}$ has as good a solution as any of the solutions which could be obtained by continuing on the current path beyond node k.

This procedure will cut down on the number of basic feasible solutions that need to be examined; its efficiency depends heavily on how good the bounds z^k are and on how close $\tilde{z}$ is to the true optimal value of z.

Any heuristic method can be used to obtain an initial $\tilde{z}$; thereafter, every time a path is terminated because a unique basic feasible solution has been obtained, the corresponding value of z, $\hat{z}$, can be calculated and compared with the current $\tilde{z}$; if $\hat{z}$ is smaller than $\tilde{z}$, then $\hat{z}$ and its corresponding x replace $\tilde{z}$ and its corresponding x as the current upper bound estimate of the optimal solution. A method for calculating z^k is given in the next section.

The branch and bound algorithm can be summarized as follows:

1. An initial upper bound, $\tilde{z}$, of the optimal value of the objective function is obtained by means of a heuristic. This bound and its corresponding solution vector x are stored.

2. The paths are then generated so that in branching from one node to another, the node at which the additional constraint is of the form "$x_j > 0$" is selected as the next node in the path, until one of five path terminating rules occurs. The first three such rules are given on page 223. Rules 1 and 2 indicate that a unique solution has been reached (although it may not be feasible). Rule 3 pertains to infeasibility; that is, no basic feasible solution exists for the constraints (13). Rule 4 is the initial bounding procedure: A path is terminated if the lower bound z^k of the objective function is as large as the current upper bound of z^*; that is, if (14) holds, then the basic feasible solution corresponding to $\tilde{z}$ has as good a solution as any of the solutions which could be obtained by continuing on the current path. Hence, the path is terminated. Rule 5 pertains to the fixed charges themselves. When the only fixed charges whose subscripts are not in either S_1^k or S_2^k are all zero, then as will be seen in the next section, the lower bound z^k is the actual minimum value of the objective function (1) subject to the constraints (13). Hence, the path can be terminated (and if $z^k < \tilde{z}$, then z^k and its corresponding solution vector become the new upper bound).

3. When the last path has been terminated, the current lower bound z and its corresponding x will be the optimal solution.

3.3 Computation of the Lower Bounds

In order to obtain the lower bound z^k some initial computations are made. In particular, it will be useful to know the level of degeneracy that exists in (2); that is, how many basic variables can appear in a basic solution at a zero level. To obtain this information, the following n linear programming problems are solved:

(15) Minimize $z_j = x_j$,
subject to: x is an extreme point of the convex set (2) and x_j is in the basis.

The method for solving these restricted linear programming problems will be presented later in this section.

If the optimal solutions to (15) are denoted by x_j^*, respectively, then if all $x_j^* > 0$, no degeneracy exists; if any $x_j^* = 0$, then the number of such variables determines the maximum level of degeneracy which could occur.

It shall be assumed that:

(16) $$d_1 \geq d_2 \geq \ldots \geq d_n.$$

In practice (16) is really no restriction, since the vectors may only have to be reordered so that (16) holds.

Let $S_3^k = \{j | j \notin S_1^k \cup S_2^k\}$,
$P = \{j | x_j$ can be in the basis at a zero level$\}$,
$Q = P \cap (S_2^k \cup S_3^k)$,
$N_1^k =$ the number of elements in S_1^k, and
$N_Q^k =$ the number of elements in Q.

Thus S_3^k is the set of subscripts whose corresponding variables do not appear as an additional constraint at node k; the set Q consists of those subscripts whose corresponding variables could appear in the basis at a zero level at node k.

Consider the following problem:

(17) Minimize $z_1 = cx$
subject to: $x \in T^k$.

The desired optimal solution to problem (17), for any node k, lies at an extreme point of the constraints (2) and also satisfies the additional constraints imposed at node k. Denote this optimal solution by z_1^k and let

(18) $$z_2^k = \sum_{j \in S_1^k} d_j.$$

Further, let z_3^k denote the value of the sum of the last $(m - N_1^k - N_Q^k)$ fixed charges. Define

(19) $$z^k = z_1^k + z_2^k + z_3^k.$$

In order for z^k to be a legitimate lower bound for z^*, it is sufficient to show that for any solution $x \in T^k$ (i.e., any solution x which satisfies the constraints at node k),

(20) $$cx + dy \geq z^k.$$

But,

(21) $$cx + dy = cx + \sum_{j \in S_1^k} d_j + \sum_{j \in S_3^k} d_j y_j,$$

since for $j \in S_1^k$, $y_j = 1$ and for $j \in S_2^k$, $y_j = 0$. By definition, $cx \geq z_1^k$.

(22)

Since the maximum level of degeneracy possible at node k is N_Q^k, there must be at least $(m-N_Q^k)$ fixed charges whose $y_j=1$ in any basic solution at node k; but, N_1^k of these fixed charges are determined by S_1^k (and are included in z_2^k). Thus, there are at least $(m-N_1^k-N_Q^k)$ remaining fixed charges whose y_j must be unity. Since z_3^k is defined to be the sum of the $(m-N_1^k-N_Q^k)$ smallest remaining fixed charges, it must be true that

$$\sum_{j\epsilon S_3^k} d_j y_j \geq z_3^k \tag{23}$$

Addition of the three relations (18), (22), and (23) yields the desired result (20). Hence z^k as defined by Eq. (19) is a legitimate lower bound for the optimal solution which could be attained at node k.

In order to complete the description of the algorithm it is only necessary to describe the procedure for solving the linear programming problems (15) and (17). Since (15) is merely a special case of (17) only the latter is considered below.

Consider the following problem:

$$\text{Minimize } z_1 = cx \tag{24}$$

$$\text{subject to: } x\epsilon T^k. \tag{25}$$

The set of extreme points which satisfy the constraints (25) is a connected set; that is, an adjacent extreme point path exists between any two points in the set. Suppose an initial basic feasible solution to (25) is known, and that a Simplex algorithm is used to improve the objective function. Then, in a finite number of steps, a stage will be reached (call it Stage One) in which the only vectors which could be entered into the basis to yield a decrease in the objective function would require one of the vectors whose corresponding $j\epsilon S_1^k$ to leave the basis. (It is assumed throughout the discussion that none of the vectors whose corresponding $j\epsilon S_2^k$ are considered for entry into the basis.) Let the current solution at this stage be denoted by $\hat{\bar{x}}$, with corresponding $\hat{z}_1 = c\hat{\bar{x}}$. It is possible that some other extreme point $\hat{\bar{x}}_0$ exists such that $z_0 < \hat{z}_1$, but that $\hat{\bar{x}}$ and $\hat{\bar{x}}_0$ are not adjacent extreme points. In order to determine whether or not $\hat{\bar{x}}$ is indeed an optimal solution to problem (17), the following constraint is appended to (25):

$$cx + x_{n+1} = \hat{z}_1. \tag{26}$$

The basis size is now increased to $m+1$, and the $(m+1)$st basic variable for the current solution is $x_{n+1}=0$. Now, however, it is possible that one or more variables may be brought into the basis without removing any of the variables whose corresponding $j\epsilon S_1^k$. If $\hat{\bar{x}}$ is not an optimal solution to (17), then a solution exists for which $x_{n+1} > 0$. In order to determine whether such a solution does in fact exist, the following test for optimality is employed:

Let R^k denote the set of variables which are not in the basis for $\hat{\bar{x}}$ (and whose $j \notin S_2^k$). Then, for each x_i in R^k, first insert x_i into the basis at a zero level, replacing x_{n+1}, and then enumerate the extreme points (of basis size $m+1$) of the convex set determined by (25) and (26), by solving

$$\begin{aligned} &\text{Maximize } z = x_i, \\ &\text{subject to: } Ax = b \\ &\qquad cx + x_{n+1} = \hat{z}_1 \\ &\qquad x_i - x_{n+2} = \hat{x}_i \\ &\qquad x \geq 0 \end{aligned} \tag{27}$$

$$x_j > 0, \qquad j \epsilon S_1^k$$

$$x_j = 0, \qquad j \epsilon S_2^k$$

$$x \text{ is an extreme point,}$$

where $\hat{x}_i$ is the current value of x_i, initially zero. At each step in solving (27) the minimum increase in x_i is taken. In this manner every extreme point of (25) and (27) which contains x_i will be examined. If at any stage $x_{n+1} > 0$, the procedure can be terminated, and by beginning with this new extreme point, the objective function (24) can be reduced in the usual way until Stage One is again reached. If, on the other hand, the optimal solution to (27) for each x_i is zero, or no extreme points for which $x_{n+1} > 0$ have been found after solving (27) for all $x_i \epsilon R^k$, then $\hat{\hat{x}}$ is the optimal solution to problem (17).

This procedure will not, in general, be as lengthy as it might at first appear. If $\hat{\hat{x}}$ is the unique optimal solution to problem (17), then the problems (27) will all have optimal values of zero, and one iteration per problem will indicate this. Moreover, if $\hat{\hat{x}}$ is not in fact optimal, is should not require very many iterations to find a solution for which $x_{n+1} > 0$.

It should also be noted that the true minimal solution to problem (17) is required only when $\hat{z}_1 > (\tilde{z} - z_2^k - z_3^k)$, where $\tilde{z}$ is the upper bound estimate, since if $\hat{z}_1 \leqslant \tilde{z}$ the path will not be terminated and decreasing $\hat{z}_1$ further is of no interest. Hence the above test for optimality for problem (17) need be applied at any node only when $\hat{z}_1 > \tilde{z}$.

The question still remains: How is an initial basic feasible solution to (17) obtained? The procedure is as follows:

$$\text{Let } W_1 = \{x_j | j \epsilon S_2^k,\ x_j \text{ not in current basis}\}$$

$$W_2 = \{x_j | j \epsilon S_2^k,\ x_j > 0 \text{ in current basis}\}.$$

Thus, those $x_j \epsilon W_1$ satisfy the constraints imposed at node k, and those $x_j \epsilon W_2$ do not; in order to remove those $x_j \epsilon W_2$ from the basis, the following linear programming problems may be solved, for each $x_j \epsilon W_2$:

$$\text{Minimize } z = x_j,$$

$$\text{subject to: } Ax = b \tag{28}$$

$$x \geqslant 0.$$

When $z = 0$, either x_j has been removed from the basis, or it is in the basis at a zero level and can be removed; if in an optimal solution $z > 0$, no feasible solution exists with x_j not in the basis.

It should be noted that once a variable $x_j \epsilon W_2$ has been removed from the basis, then $x_j \epsilon W_1$; from the theory of linear programming it is known that if it is desired that some variables not appear in a basic solution, they may merely be excluded from consideration for entry into the basis; thus, proceeding to remove the variables $x_j \epsilon W_2$ from the basis while never considering the variables $x_j \epsilon W_1$ for entry into the basis will yield in a finite number of steps a basic solution in which $W_1 = S_2^k$ and W_2 is empty.

Now, the variables $x_j \epsilon S_1^k$ must be inserted into the basis:

$$\text{Let } Y_1 = \{x_j | j \epsilon S_1^k \text{ and } x_j > 0\}$$

$$Y^2 = \{x_j | j \epsilon S_1^k \text{ and } x_j = 0\}.$$

For those variables $x_j \epsilon Y_2$ the following linear programming problems are solved:

$$\begin{aligned} &\text{Maximize} && z = x_j \\ &\text{subject to:} && Ax = b \\ &(29) && x \geq 0 \\ & && x_j \epsilon Y_1 \text{ remain in the basis at a positive level} \\ & && x \text{ is an extreme point.} \end{aligned}$$

Since the $x_j \epsilon Y_1$ are already in the basis at a positive level, the problems (29) can be solved by the method discussed earlier in this section. If the optimal solution to any of the above problems is zero, then no feasible solution to problem (29) exists with that variable in the basis at a positive level; if any solution exists for which $z > 0$, then variable x_j has successfully entered the basis (it is not necessary actually to optimize the objective function). At this point, x_j will now be in Y_1 and the next variable $x_j \epsilon Y_2$ can be inserted in the basis via problem (29). In a finite number of steps, a stage will be reached at which $Y_1 = S_1^k$ and Y_2 is empty. If none of the variables $x_j \epsilon S_2^k$ were considered for entry into the basis during the procedure of solving the problems (29), then an initial basic feasible solution to problem (17) has been obtained.

It should be noted that at no stage of the branch and bound procedure will more than one variable not satisfy the additional constraints imposed by any node k. Thus, the amount of actual computation necessary to obtain the initial solution is much less than one might at first suspect.

Moreover, the amount of computer storage required by the algorithm does not build up as the number of nodes examined increases, but remains constant throughout the procedure (for a problem of any given size). This advantage does not exist in most other branch and bound algorithms.

4. COMPUTATIONAL RESULTS

The primary difficulty in finding fairly large problems to test heuristic methods is that the optimal solution must be known. Cooper and Drebes [3] have randomly generated 290 (5×10) problems with the following properties:

(1) $|a_{ij}| \leq 20$,

(2) $1 \leq c_j \leq 20$,

(3) $1 \leq d_j \leq 999$,

(4) The density of A is 50 percent.

The optimal solutions to these problems were obtained by total enumeration.

In order to take advantage of these problems, larger problems were constructed by forming the A matrix as follows:

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & A_k \end{bmatrix}$$

where the A_i are (5×10) matrices. The resulting problem is of size $(5k \times 10k)$ and its corresponding solution is known. Hence problems of varying size can be formulated to test the heuristic methods. Because of roundoff difficulties, some of the test problems are not included in the results. These are noted under the proper table.

Since the heuristics tested are all adjacent extreme point methods, each iteration corresponds to a regular linear programming transformation. Thus, the primary means of comparison among different heuristic approaches is the number of iterations before the best obtained solution is found. Since these methods are not exact, the "best obtained solution" will not always be the optimal solution, and in some cases will be a function of the parameters related to the particular method.

The approximate computer times (for an IBM 360/50) per iteration for the four groups of problems for both Heuristic One and Heuristic Two were (in seconds): 0.10, 0.18, 0.60, and 2.4.

The Cooper and Drebes paper [3] reports computational experience with two heuristics: both heuristics were tested on a set of 253–(5×10) problems; 240 were solved exactly by the first method in an average of 75 total iterations per problem, and 245 problems were solved exactly by the second method in an average of 100 total iterations per problem. The three (5×10) problems which neither heuristic solved exactly were all less than 10 percent from optimality. No information on the number of iterations required to achieve the best solution is given.

On a set of 70–(15×30) problems, the two methods were combined; the resulting heuristic solved 63 problems exactly with an average of 1200 iterations per problem. The nonoptimal problems were less than 3 percent from optimality.

4.1 Heuristic One

Problems of size (5×10), (15×30), (30×60), and (50×100) were used in the testing of Heuristic One. Of the 250–(5×10) problems, 235 were solved exactly with $\alpha_0 = 2$ and $\beta_0 = 2$. The average number of iterations required to achieve the final solution was 3 and the average total number of iterations per problem was 6.8. Table 1 contains a summary of the 15 nonoptimal problems. When α_0 was increased to 3 and β_0 to 4, problems 2, 94, and 128 were solved exactly. No further increase in α_0 and β_0 yielded any improvements in the 12 remaining suboptimal problems.

Ninety (15×30) problems were attempted, and with $\alpha_0 = 2$, $\beta_0 = 2$, the optimal solutions to 75 problems were obtained, with an average of about 5 iterations before the best solution was obtained and an average of 12 total iterations per problem. No further improvement resulted when α_0 was increased to 10; the average total number of iterations increased to 16. β_0 was increased again to 15, with an improvement in only one of the suboptimal solutions-problem 43. The optimal solution is $Z^* = 8317.7$ and the best previously obtained solution was $Z = 9172.4$; with $\beta_0 = 15$, the solutions 8964.7, 8593.0, and 8525.8 were obtained. The average total number of iterations was 23. Table 2 contains a summary of the suboptimal (15×30) problems.

TABLE 1. *Heuristic One-(5×10) Suboptimal Problems*

Problem number[a]	Optimal solution	Best obtained solution	Percent from optimal
2	2830.4	[b] 3317.6	14.7
57	4015.8	4017.9	0.05
70	3363.0	3468.3	3.1
86	2359.4	2380.7	0.9
93	2351.7	2436.6	3.6
94	2373.0	[b] 2899.1	1.1
99	2418.0	2473.0	2.3
128	3459.0	[b] 3470.6	0.3
131	1543.0	1766.1	14.5
137	3593.1	3885.1	8.1
181	2826.2	3183.0	12.6
188	2417.1	2440.1	0.95
235	2080.9	2155.9	3.6
267	3072.2	3170.9	3.2
279	2323.4	2533.7	9.1

[a] Problems not attempted: 29, 36, 51, 55, 66, 67, 73, 75, 82, 88, 98, 117, 119, 120, 126, 130, 134, 141, 147, 158, 162, 164, 166, 176, 185, 201, 204, 206, 211, 215, 219, 226, 236, 237, 238, 248, 250, 260, 264, 266, 275.

[b] Exact solutions obtained with $\alpha_0 = 3$ and $\beta_0 = 4$.

TABLE 2. *Heuristic One-(15×30) Suboptimal Problems*

Problem number[a]	Optimal solution	Best obtained solution	Percent from optimal
20	7676.4	7726.9	0.67
29	7522.6	7544.0	0.28
30	9452.9	9458.2	0.06
31	9201.7	9286.5	0.89
32	6666.4	7192.6	7.89
33	8738.3	8793.3	0.63
40	11,117.3	11,365.9	2.23
43	8317.7	8525.8	2.50
45	6785.6	6938.3	2.25
46	8517.7	8809.7	3.43
53	6497.6	6647.7	2.31
61	7651.0	8007.7	4.66
63	7836.0	7859.6	0.29
85	6307.2	6635.2	5.20
93	8592.7	9047.5	5.29

[a] Problems not attempted: 1, 44, 64, 68, 79, 89.

Thirty-six (30×60) problems were attempted and 26 were solved optimally with $\alpha_0 = 6$ and $\beta_0 = 25$. The average number of iterations before the best solution was obtained was about 11 iterations and the total number of iterations per problem averaged about 42. A summary of the suboptimal (30×60) problems is given in Table 3.

TABLE 3. *Heuristic One-(30 × 60) Suboptimal Problems*

Problem number [a]	Optimal solution	Best obtained solution	Percent from optimal
15	16,975	17,002	0.04
16	15,868	16,479	3.98
17	18,441	18,496	0.30
23	15,303	15,748	2.91
25	16,886	16,987	0.50
27	13,960	14,110	1.07
31	14,944	15,301	2.39
32	17,350	17,440	0.52
37	14,716	15,181	3.16
43	13,946	14,274	2.35

[a] Problems not attempted: 1, 5, 10, 20, 22, 24, 34, 40, 45.

Nine-(50×100) problems were also tested, with $\alpha_0 = 10$ and $\beta_0 = 40$; the results are given in Table 4.

TABLE 4. *Heuristic One-(50 × 100) Problems*

Problem number	Optimal solution	Best obtained solution	Percent from optimal	No. of iterations to best obtained solution	Total No. of iterations
1	26,761	26,761	.....	47	87
2	28,842	28,842	.....	22	62
5	25,545	25,545	.....	21	61
12	29,350	30,182	2.83	14	49
14	24,444	25,085	2.57	28	46
15	29,380	29,481	0.34	13	48
18	26,844	26,844	.....	18	51
19	25,903	26,350	1.73	24	49
20	27,822	27,899	0.28	18	48
Average No. of Iterations				22.8	55.7

4.2 Heuristic Two

The same sets of problems used to test Heuristic One were employed in the testing of Heuristic Two. In some cases, again, roundoff difficulties caused some problems to be discarded, and these problems are again noted in the tables.

Of the 268–(5×10) problems attempted by Heuristic Two, 255 were solved optimally. A summary of the suboptimal problems appears in Table 5. The average number of iterations required to achieve the final solution was 6, and the average total number of iterations was 15.2.

TABLE 5. *Heuristic Two-(5 × 10) Suboptimal Problems*

Problem number[a]	Optimal solution	Best obtained solition	Percent from optimal
28	2993.4	3025.4	1.1
50	2383.9	2813.4	18.0
71	3058.3	3470.5	13.5
93	2351.7	2436.6	3.6
99	2418.0	2473.0	2.3
108	2414.8	2423.8	0.4
119	3467.6	3560.6	2.7
120	3579.1	3734.5	4.3
131	1543.0	1766.2	14.5
137	3593.1	3885.1	8.1
154	2589.5	2622.7	1.3
245	3574.7	3635.8	1.7
279	2323.4	2533.7	9.1

[a] Problems not attempted: 23, 27, 30, 52, 61, 67, 76, 81, 94, 98, 110, 128, 149, 158, 165, 202, 210, 224, 225, 239, 249, 254.

Of the 287–(5×10) problems attempted by at least one of Heuristic One and Heuristic Two, 280 were solved optimally; the following 7 problems were not solved optimally by either method: 93, 99, 119, 120, 131, 137, and 279.

Eighty-four (15×30) problems were attempted by Heuristic Two, and 74 were solved optimally. The average number of iterations required to obtain the final solution was 12, and the average total number of iterations per problem was 43.

TABLE 6. *Heuristic Two-(15 × 30) Suboptimal Problems*

Problem number[a]	Optimal solution	Best obtained solution	Percent from optimal
1	8096.2	8142.6	0.57
31	9201.7	9286.5	0.89
36	6162.5	6171.6	0.15
40	11,117.5	11,365.9	2.23
43	8525.8	8537.4	2.50
44	6807.3	6905.5	1.44
46	8517.7	8809.7	3.43
52	7880.0	8214.1	2.97
79	7765.0	7840.0	2.45
93	8592.7	8803.0	5.29

[a] Problems not attempted: 16, 18, 21, 22, 26, 27, 33, 37, 61, 72, 80, 83.

Of the 96 (15 × 30) problems attempted by at least one of Heuristic One and Heuristic Two, only the following 5 problems were not solved optimally: 31, 40, 43, 46, and 93.

TABLE 7. *Heuristic Two-(30 × 60) Suboptimal Problems*

Problem number [a]	Optimal solution	Best obtained solution	Percent from optimal
1	15,490	15,536	0.30
20	18,407	18,655	1.34
22	15,333	15,535	1.32
23	15,303	15,595	1.91
26	16,848	17,346	2.91
29	15,773	15,889	0.73
34	16,912	17,076	0.97
41	15,601	15,662	0.31
45	14,243	14,267	0.17
47	15,852	16,062	1.32

[a] Problems not attempted: 5, 9, 11, 13, 14, 16, 17, 19, 31, 36, 40, 42, 43.

Thirty-six (30 × 60) problems were attempted by Heuristic Two, and 26 were solved optimally. The average number of iterations required to obtain the final solution was 25, and the average total number of iterations per problem was 81.

Of the 46 (30 × 60) problems attempted by at least one of Heuristic One and Heuristic Two, 42 were solved optimally; the following problems were not solved optimally by either method: 1, 23, 34, and 45.

Heuristic Two was not tested on any larger problems.

4.3 The Branch and Bound Method

Fifteen (5 × 10) problems and 10 (15 × 30) problems were solved by the branch and bound procedure. The results are given in Tables 8 and 9. The total possible number of nodes that might have to be

TABLE 8. *The Branch and Bound Algorithm-(5×10) Problems*

Problem number	Number of nodes	Time (minutes)
1	66	0.23
2	24	0.16
3	20	0.16
4	44	0.26
5	20	0.13
6	14	0.11
7	20	0.13
8	74	0.31
9	10	0.09
10	38	0.23
11	34	0.17
12	12	0.10
13	40	0.16
14	24	0.13
15	42	0.30
Average	32	0.16

TABLE 9. *The Branch and Bound Algorithm-(15×30) Problems*

Problem number	Number of nodes	Time (minutes)
6	1242	17.9
8	672	9.1
9	3144	47.3
14	1134	18.3
18	308	8.5
29	2294	43.0
34	502	26.4
35	334	4.8
37	904	11.6
38	1546	24.4
Average	1208	21.1

examined is $2^{n+1} - 1$. The average number of nodes that were examined was 32 and 1208, for the two groups of problems, respectively. Also included in the tables is the computer time required to move through the nodes; these times are for an IBM 7072. It is felt that these times could be greatly improved if the computer code were made more efficient. There are many improvements that could be made in the code, such as rewriting it in machine language and utilizing a more efficient Simplex method (such as revised Simplex).

The average number of nodes examined for the (5×10) problems was 2^5 out of a possible total of approximately 2^{10} nodes; for the (15×30) problems the average number of nodes actually examined was approximately 2^{10} out of a possible total of approximately 2^{30}. Hence, the ratio of the exponents of the number of nodes examined to the number of total possible nodes decreased when the number of constraints was tripled (from 5 to 15). If it is assumed that this ratio continues to decrease linearly, then the number of nodes that would have to be examined for a problem of size (45×90) is about 2^{20} nodes.

Hence, it appears that the number of nodes examined grows rapidly with the number of variables, and so the efficiency of the computer code is extremely important.

5. SUMMARY AND CONCLUSIONS

The computational results of Section 4 clearly indicate that the heuristic algorithms are orders of magnitude faster than the branch and bound algorithm. Moreover, the heuristics almost always provide very good, if not optimal, solutions. When two or more heuristics are combined, the likelihood of finding the optimal solution increases considerably. In fact, when the two heuristics of Cooper and Drebes [3] are combined with Heuristic One and Heuristic Two, all 290 of the (5×10) problems were solved optimally, only one problem of the 96 (15×30) problems (problem 53) was not optimally solved.

The branch and bound algorithm presented herein is a new, exact algorithm for the solution of the fixed charge problem and appears to be much better than the mixed integer-continuous variable approach discussed in Section 1.2. Such an exact method of solution is essential for those situations in which the true optimal solution must be determined.

It is felt that many improvements could be made in the branch and bound algorithm, in addition to refinements in the computer code used to test the algorithm. It is hoped that a better method of

determining the true maximum level of degeneracy of the convex set of solutions can be found, so that a better lower bound for the objective function at any given node may be computed. Other refinements in the handling of the degeneracy problem could probably also be found.

It is suggested that research be conducted on the problem of generalizing the branch and bound approach described herein, so that it would be applicable to more general separable concave functions. Another generalization which should be considered is the case in which the fixed charges are dependent upon more than one variable. For example, a fixed charge d_{ij} could be incurred only if the corresponding variables x_i and x_j are both positive.

6. ACKNOWLEDGMENTS

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* * *

ON COMPUTING THE EXPECTED DISCOUNTED RETURN IN A MARKOV CHAIN*

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ABSTRACT

The discounted return associated with a finite state Markov chain $X_1, X_2 \ldots$ is given by $g(X_1) + \alpha g(X_2) + \alpha^2 g(X_3) + \ldots$, where $g(x)$ represents the immediate return from state x. Knowing the transition matrix of the chain, it is desired to compute the expected discounted return (present worth) given the initial state. This type of problem arises in inventory theory, dynamic programming, and elsewhere.

Usually the solution is approximated by solving the system of linear equations characterizing the expected return. These equations can be solved by a variety of well-known methods. This paper describes yet another method, which is a slight modification of the classical iterative scheme. The method gives sequences of upper and lower bounds which converge monotonely to the solution. Hence, the method is relatively free of error control problems.

Computational experiments were conducted which suggest that for problems with a large number of states, the method is quite efficient. The amount of computation required to obtain the solution increases much slower with an increase in the number of states, N, than with the conventional methods. In fact, computational time is more nearly proportional to N^2, than to N^3.

I. Introduction

Consider a Markov chain $X_1, X_2, \ldots$, taking values in the finite set $S = [1,2, \ldots, N]$ that has transition probabilities $p(y;x) = Pr[X_{t+1} = y | X_t = x]$. When in state x an amount $g(x)$ is received immediately. The discounted return associated with passing through the sequence of states $X_1, X_2, \ldots$, is

$$g(X_1) + \alpha g(X_2) + \alpha^2 g(X_3) + \ldots,$$

where α is the discount factor ($0 \leq \alpha < 1$).

The expected discounted return given $X_1 = x$, represented by $u(x)$ (also called the present worth of state x), is well known to be the unique solution of the equation

$$u(x) = g(x) + \alpha \sum_{y \in s} u(y) p(y;x). \tag{1}$$

An efficient machine algorithm for solving (1) is desired.

The problem of computing the expected discounted return frequently arises in inventory problems, replacement problems, and elsewhere (c.f. [3], [4]).

If N (the number of states in S) is reasonably small, then (1) can be solved by familiar techniques, such as Gausian elimination or the simplex method, and is a straight forward problem in machine computation. However, if there is a large number of states, the amount of computation, which is generally

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proportional to N^3 (cf.[1, p. 215]) using such methods, may be prohibitively expensive. Also, with these methods error control problems can arise even in problems of moderate size.

This note describes some computational experiments using a slightly modified version of the classical iterative method for computing u. The method determines sequences of functions $\{u_n'\}$ and $\{u_n''\}$ on S which converge monotonely to u from below and above, respectively. Then $\hat{u}_n = (u_n' + u_n'')/2$ is taken as the estimate of u. The experiments indicate that for "large" problems (say $N \geqslant 100$), the amount of computation required to obtain a solution whose percentage error at each state is less than a preassigned figure, increases much more slowly than N^3. For example, in a number of the experiments with $N = 500$, approximately $16(500^2)$ elementary (multiply and add) computations were required to achieve 0.005-percent accuracy. As with the estimate produced by the classical iteration scheme (see Eq. (5) below) convergence is found to be exponential, but at a rate significantly faster than the classical method. The monotonely convergent upper and lower bounds, which are easily calculated, provide a very satisfactory means of controlling error due to round-off, which may accumulate with other methods.

The method starts with any function v_1 which is 0 at a selected state $s \epsilon S$, with the functions v_2, v_3, . . ., being defined by

$$v_{n+1}(x) = g(x) + \alpha \sum_y v_n(y)p(y; x) - \left(g(s) + \alpha \sum_y v_n(y)p(y; s)\right). \tag{2}$$

Thus $v_n(s) = 0$, $n = 1, 2, \ldots$. Let

$$\begin{aligned} L_n' &= \min_x \left[g(x) + \alpha \sum_y v_n(y)p(y; x) - v_n(x)\right], \\ L_n'' &= \max_x \left[g(x) + \alpha \sum_y v_n(y)p(y; x) - v_n(x)\right]. \end{aligned} \tag{3}$$

The sequences $\{u_n'\}$ and $\{u_n''\}$ are then given by

$$\begin{aligned} u_n' &= v_n + (1 - \alpha)^{-1}L_n' \\ u_n'' &= v_n + (1 - \alpha)^{-1}L_n''. \end{aligned} \tag{4}$$

The sequences $\{u_n'\}$ and $\{u_n''\}$ have the properties:

$$\text{(i) } u_n' \leqslant u \leqslant u_n'', \quad \text{(ii) } u_n' \leqslant u_{n+1}', \quad u_n'' \geqslant u_{n+1}'', \quad \text{and} \quad \text{(iii) } u_n' \to u, \quad u_n'' \to u.$$

These properties are obtained directly as special cases of Theorem 2 in [2].

The modification represented by the calculation of $\hat{u}_n$ is to be compared with the classical iterative method. The latter is defined by an initial function u_1 and the rule

$$u_{n+1}(x) = g(x) + \alpha \sum_y u_n(y)p(y; x). \tag{5}$$

If $u_1(x) = v_1(x) + u_1(s)$, then $v_n(x) = u_n(x) - u_n(s)$, and

$$\hat{u}_n(x) = u_n(x) + \tfrac{1}{2}(1 - \alpha)^{-1}(J_n' + J_n''), \tag{6}$$

where

$$\begin{aligned} J_n' &= \min_x \left[g(x) + \alpha \sum_y u_n(y)p(y; x) - u_n(x)\right], \\ J_n'' &= \max_x \left[g(x) + \alpha \sum_y u_n(y)p(y; x) - u_n(x)\right]. \end{aligned} \tag{7}$$

II. Experimental Procedure

To explore the computational efficiency of the suggested method, it was decided to investigate four main types of transition matrices, with N equal to 10, 50, 100, 500, or 1,000. The discount factor was varied over the four values 0.90, 0.95, 0.99, and 0.995 representing the likely range of applications. In all the experiments, the initial function v_1 was identically zero.

The four main types of transition matrices were as follows:
(I) *Random.* In this type of matrix, the row entries were randomly generated by selecting for each entry a uniformly distributed random number between 0 and 1 and then normalizing by dividing by the row total.
(II) *Large probability in the left hand column.* Here the first column has a specified value p_0 (i.e., $p(1; x) = p_0$ for $x = 1, 2, \ldots, N$). The value of p_0 was varied over the three values 0.1, 0.5, and 0.9, with the remaining entries generated randomly as above (summing to $1 - p_0$, of course).
(III) *Large probability on the diagonal.* Here all diagonal elements were set equal to p_0, and the other entries were random as before. The value of p_0 was varied as above.
(IV) *Large probabilities scattered.* In this type of matrix, the first row was generated by taking four probabilities $p_1 = p_0/2$, $p_2 = p_0/4$, $p_3 = p_0/8$, and $p_4 = p_0/8$ and spacing them evenly in the first row. For example, for $N = 100$ we have $p(1; 1) = p_1$, $p(25;1) = p_2$, $p(50; 1) = p_3$, and $p(75;1) = p_4$. The remaining rows of $p(y;$ x) were obtained from the first row by cyclic permutations, the last entry in the first row becoming the first entry in the second row, etc.

It was decided arbitrarily to have $g(x)$ vary in multiples of 5 as follows: for $n = 10$, $g(1) = 5$, $g(2) = 10, \ldots, g(10) = 50$. For the remaining values of N, all of which were multiples 10, the first $N/10$ values of g were 5, the second $N/10$ values were 10, etc., with the last $N/10$ values being 50. Evidence that this method of choosing g, while arbitrary, was not malicious, was obtained by repeating all the experiments for $N = 100$ for matrices of Type III (large probabilities on the diagonal) with a different scheme for choosing g. Each value of $g(x)$, $x = 1, 2, \ldots, N$, was determined as a uniformly distributed random number between 0 and 100. Matrices of Type III were chosen for this additional experimentation since they were the most difficult of the four in terms of convergence times. The number of iterations (see below, Table 1) using the randomly determined g was essentially identical to those obtained with the fixed g, differing by 0, 1 or at most 2 iterations.

III. Results

The main results are presented in Table 1. The entries in the table represent the number of iterations (the value of n in Eq. (4)) required for the maximum possible percentage error (*MPPE*) of the estimate $\hat{u}_n$ to be less than 0.005 or 0.5 percent. Thus, for the indicated value of n, $\max_x |u(x) - \hat{u}_n(x)|/u(x)$ is less than 0.00005 or 0.005. The upper and lower entries in each cell in the table correspond respectively to the 0.005 and 0.5 percent figures. The values of n were actually computed by allowing the computer program to run until $\max_x (u_n''(x) - u_n'(x))/u_n'(x) \leq 0.0001$, which implies $\max_x |u(x) - \hat{u}(x)|/u(x) \leq 0.00005$. The 0.5-percent values were then determined by inspection of the output which included the *MPPE* for each iteration.

It should be noted that the *MPPE* may be considerably larger than the actual percentage error. As a typical example, consider the matrix of Type III with $P_0 = 0.90$, $N = 100$ and $\alpha = 0.90$ where the *MPPE* after 16 iterations was 6.4 percent. However, $\hat{u}_{16}(x)$ differed by at most 1.4 percent from $\hat{u}_{52}(x)$, and the latter (which corresponds to the last iteration performed) has at least 0.005 percent accuracy.

The *MPPE* would not be a meaningful criterion of accuracy if $u(x)$ were very near zero for some x. This possibility was avoided by the choice of strictly positive values of g. The *MPPE* can also be rendered meaningless by adding a large positive constant to each value of g, since if u is the solution with immediate return function g, $u+A/(1-\alpha)$ is the solution with immediate return $g+A$. Thus by making A sufficiently large, the *MPPE* in the approximate and easily computed estimate, $\hat{u}=A/(1-\alpha)$ can be made arbitrarily small. Again, the choice of g avoids this trivial extreme.

Several general findings were noted. First, the *MPPE* exhibited a strong tendency to decrease exponentially with the number of iterations, i.e., as $kr^n, 0 > r > 1$. Moreover k is typically much less than unity. This was clearly evident on plotting the error against n on semi-logarithmic paper. The maximum percentage error for the classical method defined by equation (5), also decreases exponentially, approximately as $K\alpha^n$ where K is near unity. However, r was appreciably less than α, the ratio r/α being .9 or less in all the experiments. Thus the ratio of percentage errors for the two methods after 30 iterations, say, would be around .05 and perhaps smaller.

A second finding is that the method automatically takes advantage of certain simplifying features of the transition matrices. Thus in matrices of Type I and II, the distribution of the next value of g is very similar from state to state, and the sequence $g(X_1)$, $g(X^2)$, . . ., is a sequence of similarly distributed, nearly independent random variables. For that reason, a good estimate for $u(x)$ is simply the immediate return $g(x)$ plus the mean value of g times $\alpha/(1-\alpha)$, and a good computational scheme ought to take advantage of this simplicity. The experimental results show that the method does. Somewhat surprisingly, when the rows of the transition matrix are identical, then

$$u_2'(x) = u_2''(x) = g(x) + \alpha(1-\alpha)^{-1} \sum_y g(y)p(y; x) = u(x),$$

and the exact result is obtained immediately (given $v_1 = 0$).

This suggests a general rule, that the less the effect of the initial value of x on the sequence of ensuing returns, the less the amount of computation required. On this basis, one would predict the Type IV matrix to be easier to solve than the Type III matrix, which in fact is the case.

A third result suggested by Table 1 is that the amount of computation required to achieve a given *MPPE* increases approximately as N^2. There are approximately N^2 elementary computations (multiply and add being counted as an "elementary computation") per iteration, so that the amount of computation required in n iterations is proportional to nN^2. As is evident in Table 1, for a given combination of matrix type, P_0 and α, n increases slowly and at a decreasing rate, or does not increase at all, as N is increased. The worst case from the point of view of the relative increase in computation with N is the Type IV matrix with $\alpha = 0.95$ and $p_0 = 0.9$. However, even in this instance only nine more iterations are required to achieve 0.5-percent error as N is increased by a factor of 10 from 50 to 500. Note, however, that in Table 1 with $N = 10$, there are a good number of instances where more than 10 iterations and hence more than N^3 elementary computations are required. Thus the relative advantage of the method as against conventional method requiring approximately N^3 elementary computations, would appear to hold only for large values of N.

Finally it is to be noted that the suggested method seems to be insensitive in an appropriate way to near zero entries in the transition matrices. This is perhaps a major reason for the rapid convergence. Type I matrices for which 0.005-percent accuracy was achieved in four iterations, particularly illustrate this result as it is obvious that for $N = 500$ or $N = 1{,}000$ the entries are extremely small.

TABLE I. *Iterations Required to Achieve 0.5 and 0.005 percent Maximum Possible Error*

Matrix Type																									
I							II								III						IV				
α \ N	10	50	100	500	1000	Prob P_0	α \ N	10	50	100	500	1000		α \ N	10	50	100	500	1000		α \ N	50	100	500	1000
0.995	7 4	5 4	5 3	4 3	4 3	0.9	0.995	4 3	4 3	4 3	4 3	.....		0.995	83 46	90 51	91 51	 49	.....		0.995	58 30	61 32	.....	.....
0.99	7 4	5 4	5 3	4 3	4 3		0.99	4 3	4 3	4 3	4 3	.....		0.99	80 45	87 49	87 49	.....	.....		0.99	43 19	54 24	.....	.....
0.95	7 4	5 4	5 3	4 3	4 3		0.95	4 3	4 3	4 3	4 3	3 3		0.95	65 37	66 38	69 39	 37	.....		0.95	36 16	43 20	 25	.....
0.90	6 4	5 4	5 3	4 3	4 3		0.90	4 3	4 3	4 3	3 3	3 3		0.90	50 28	52 30	52 29	 29	.....		0.90	25 12	37 17	 20	.....
						0.5	0.995	5 3	5 3	5 3	6 3	.....		0.995	15 9	16 9	16 9	.....	.....		0.995	15 9	19 10	.....	.....
							0.99	5 3	5 3	5 3	5 3	.....		0.99	15 9	16 9	16 9	.....	.....		0.99	11 6	12 7	.....	.....
							0.95	5 3	5 3	5 3	5 3	4 3		0.95	15 8	15 9	16 9	 9	.....		0.95	11 6	12 7	12 7	.....
							0.90	5 3	5 3	5 3	4 3	4 3		0.90	15 8	15 9	14 9	 8	.....		0.90	10 6	11 6	12 7	.....
						0.1	0.995	7 4	6 3	6 3	5 3	.....		0.995	7 4	6 4	6 4	.....	.....		0.995	6 4	6 3	.....	.....
							0.99	7 4	6 3	6 3	5 3	.....		0.99	7 4	6 4	6 4	.....	.....		0.99	6 4	5 3	.....	.....
							0.95	7 4	5 3	5 3	5 3	4 3		0.95	7 4	6 4	6 4	 4	.....		0.95	6 4	5 3	5 3	.....
							0.90	7 4	5 3	5 3	4 3	3 3		0.90	7 4	6 4	6 4	 4	 4		0.90	5 3	5 3	5 3	4 3

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MORE ADO ABOUT ECONOMIC ORDER QUANTITIES (EOQ)

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ABSTRACT

This paper is concerned with the determination of explicit expressions for economic order quantities and reorder levels, such that the cost of ordering and holding inventory is minimized for specific backorder constraints. Holding costs are applied either to inventory position or on-hand inventory, and the backorder constraint is considered in terms of the total number of backorders per year or the average number of backorders at any point in time. Through the substitution of a new probability density function in place of the normal p.d.f., explicit expressions are determined for the economic order quantities and the reorder points. The resulting economic order quantities are independent of all backorder constraints. It is also concluded that under certain conditions, the minimization of ordering costs and inventory holding costs (applied to inventory position), subject to a backorder constraint, is equivalent in terms of reorder levels to minimization of the safety level dollar investment subject to the same backorder constraint.

INTRODUCTION

We are concerned with a single-echelon, multi-item, continuous review inventory system in which the process generating demands does not change with time. When an item's reorder level is reached, an order is placed. After the elapse of a leadtime, which is assumed constant, the entire order is received. For each item, it is desired to determine the constant reorder quantity and reorder point which will minimize (summing all items in the inventory) the holding and procurement costs subject to an overall backorder constraint. The holding cost may be a function of either on-hand inventory or inventory position. We will constrain total units backordered or average number of units in a backorder position, which, as is well known, is also the unit-years backordered per year. In either case, the backorders may be weighted by item essentiality.

MODEL DERIVATION

Definitions

The following definitions apply throughout the discussion:

Q: order quantity (units)
r: reorder level (units)
D: annual demands (units)
k: safety factor
σ: standard deviation of unit demand during a leadtime
a: holding cost factor
A: ordering cost (dollars per order)
c: item cost (dollars per unit)
μ: mean leadtime demands
Z: item essentialtiy (relative military worth)

x: units demanded during leadtime

η: constraint—total essentiality-weighted units backordered

β: constraint—expected number of essentiality-weighted units in a backordered position at any point in time

τ: leadtime.

We also require:

i) $r = \mu + k\sigma$

ii) $f(x)$, the probability density function (p.d.f.) describing the distribution of units demanded during the leadtime:

$$\frac{\sqrt{2}}{2\sigma} \exp\left(-\sqrt{2}\left|\frac{x-\mu}{\sigma}\right|\right) \text{ for all } x. \tag{1}$$

It is easily shown that this implies:

$$Pr[x > \mu + k\sigma] = 0.5 \exp(-\sqrt{2}k) \text{ for } k \geq 0 \tag{2}$$

and

$$Pr[x > \mu + k\sigma] = 1 - 0.5 \exp(\sqrt{2}k) \text{ for } k < 0. \tag{3}$$

The form of $f(x)$ was inspired by the work of L. L. Parker [3] who essentially assumed demands were normally distributed, and then approximated the complementary cumulative function by a function of the form ae^{-bx}. We originally chose a and b such that the mean absolute deviation from the normal distribution over the range $0 \leq k \leq 3$ was minimized ($a=0.57$, $b=1.35$). We now consider, however, that $a=0.5$ and $b=\sqrt{2}$ is more appropriate.* In addition to defining a p.d.f. that is symmetric about the mean (μ), these values also cause $Pr\left[k_1 < \frac{x-\mu}{\sigma} < k_2\right]$ to be independent of μ and σ (a desirable attribute also shared by the normal p.d.f.). Further, with these values, the expected backorders at a random point in time for $k=0$ is the same as if demands were normally distributed, for large Q/σ. This similarity was considered desirable since we are primarily interested in average backorders (rather than total backorders) and we will often have no safety level.

The value of $Pr\left[k \leq \frac{x-\mu}{\sigma} \leq k+0.2\right]$ for $k=0$ to 4.0 is given in Table 1 for both the normal p.d.f. and $f(x)$.

We will now operate as if

$$f(x) = \frac{\sqrt{2}}{2\sigma} \exp\left(-\sqrt{2}\left(\frac{x-\mu}{\sigma}\right)\right)^{**}$$

*We are indebted to Mr. Alan Kaplan of the AMC Inventory Research Office who suggested that we reconsider our values for a and b.

**The absolute value of $\frac{x-\mu}{\sigma}$ is omitted.

TABLE 1
Comparison of f(x) *to the Normal p.d.f.*

K	A^a	B^b	$C_n(x)^c$	$C_f(x)^c$
0	0.0793	0.1232	0.0793	0.1232
0.2	0.0761	0.0928	0.1554	0.2160
0.4	0.0703	0.0700	0.2257	0.2860
0.6	0.0624	0.0527	0.2881	0.3387
0.8	0.0532	0.0397	0.3413	0.3784
1.0	0.0436	0.0299	0.3849	0.4084
1.2	0.0343	0.0226	0.4192	0.4310
1.4	0.0260	0.0170	0.4452	0.4480
1.6	0.0189	0.0128	0.4641	0.4610
1.8	0.0132	0.0097	0.4773	0.4704
2.0	0.0088	0.0073	0.4861	0.4777
2.2	0.0057	0.0055	0.4918	0.4832
2.4	0.0035	0.0041	0.4953	0.4874
2.6	0.0021	0.0031	0.4974	0.4905
2.8	0.0013	0.0024	0.4987	0.4928
3.0	0.0006	0.0018	0.4993	0.4946
3.2	0.0004	0.0013	0.4997	0.4959
3.4	0.0001	0.0010	0.4998	0.4969
3.6	0.0001	0.0008	0.4999	0.4977
3.8	0.0001	0.0006	0.5000	0.4982
4.0	0.0000	0.0004	0.5000	0.4987

[a] $A = Pr\left[k \leq \dfrac{x-\mu}{\sigma} \leq k+0.2\right]$ for $n(x) = \dfrac{1}{\sigma\sqrt{2\pi}} \exp\left(-\dfrac{(x-\mu)^2}{2\sigma^2}\right)$.

[b] $B = Pr\left[k \leq \dfrac{x-\mu}{\sigma} \leq k+0.2\right]$ for $f(x) = \dfrac{\sqrt{2}}{2\sigma} \exp\left(-\sqrt{2}\left|\dfrac{x-\mu}{\sigma}\right|\right)$.

[c] $C(x) = Pr\left[0 \leq \dfrac{x-\mu}{\sigma} \leq k+0.2\right]$ for $n(x)$ and $f(x)$, respectively.

Notice that this is strongly biased against negative k. In fact, by operating as if the probability distribution could be described by Eq. (4), we are, in effect, saying that $\lim_{k \to -\infty} Pr[x > \mu + k\sigma] = \infty$, a somewhat preposterous allegation. Nonetheless, this bias is considered desirable since in actual practice, we restrict $k \geq 0$.*

Expected Backorders

In order to determine the expected number of units backordered per year (B_N) and the average number of units in a backorder status at a random point in time (B_T), we need $g(y)$ — the p.d.f. for the number of backorders at a random point in time. The following technique for obtaining $g(y)$ was developed by Hadley and Whitin [1].

$g(y)dy$ = probability that the number of units backordered is between y and $y + dy$, and

$$\frac{\sqrt{2}}{2\sigma} \exp\left(-\sqrt{2}\left(\frac{p+y-\mu}{\sigma}\right)\right) dy$$

= the probability that the number of units backordered at time, t, is between y, $y+dy$ given that the inventory position (on-hand plus due-in minus due-out) is between p, $p+dp$ at time $t-\tau$. But the probability that the inventory position is between p, $p + dp$ is dp/Q.

Therefore,

$$g(y)dy = \int_r^{r+Q} \frac{\sqrt{2}}{2\sigma} \exp\left(-\sqrt{2}\left(\frac{p+y-\mu}{\sigma}\right)\right) dy \frac{dp}{Q} \tag{5}$$

and

$$g(y) = \frac{1}{Q} \int_r^{r+Q} \frac{\sqrt{2}}{2\sigma} \exp\left(-\sqrt{2}\left(\frac{p+y-\mu}{\sigma}\right)\right) dp \tag{6}$$

and finally

$$g(y) = \frac{0.5}{Q} \exp\left(-\sqrt{2}\left(\frac{y+k\sigma}{\sigma}\right)\right) (1 - \exp(-\sqrt{2}Q/\sigma)). \tag{7}$$

Consequently, the probability of a stockout (P_{out}) is

$$\int_0^\infty g(y)dy = \left(\frac{0.5}{\sqrt{2}}\right) \frac{\sigma}{Q} (1 - \exp(-\sqrt{2}Q/\sigma)) \exp(-\sqrt{2}k). \tag{8}$$

Therefore,

$$B_N = DP_{out} = \left(\frac{0.5}{\sqrt{2}}\right) \frac{D\sigma}{Q} (1 - \exp(-\sqrt{2}Q/\sigma)) \exp(-\sqrt{2}k) \tag{9}$$

(see Ref. [1], page 178)

and

$$B_T = \int_0^\infty yg(y)dy = \frac{0.5}{2} \frac{\sigma^2}{Q} (1 - \exp(-\sqrt{2}Q/\sigma)) \exp(-\sqrt{2}k). \tag{10}$$

Other Inputs

We must now obtain the average number of procurements per year, the expected inventory position ($\acute{E}[IP]$), and the expected on-hand assets ($E[OH]$). Clearly, the average number of procurements is D/Q.

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*The nonnegativity requirement for k is a result of the desire to make this model acceptable to practicing logisticians.

It is also well known that

$$E[\text{Due-in}] = \mu \qquad \text{(see Ref. [1], page 187)}$$

and

$$E[IP] = r + Q/2 = \mu + k\sigma + Q/2.$$

By definition,

$$E[IP] = E[OH] + E[\text{Due-in}] - E[B/0].$$

Therefore,

$$\begin{aligned} E[OH] &= E[IP] - E[\text{Due-in}] + E[B/0] \\ &= \mu + k\sigma + Q/2 - \mu + B_T \\ &= k\sigma + Q/2 + B_T. \end{aligned}$$

FOUR INVENTORY MODELS

We are now ready to proceed to the four models which will enable us to determine Q and r for each item in such a way as to minimize the holding cost (applied either to $E[OH]$ or $E[IP]$) plus the procurement cost, subject to an overall backorder constraint. This constraint may be either the total units backordered per year (i.e., the stockout penalty is independent of the duration of the stockout) or the average number of units in a backorder position (i.e., the stockout penalty depends on the duration of the stockout). In either case, the backorders may also be weighted by item essentiality.

MODEL I: Stockout penalty independent of duration of stockout; holding cost applied to on-hand inventory. We wish to minimize

$$\text{(I.1)} \qquad \sum_{i=1}^{n} \frac{A_i D_i}{Q_i} + \sum_{i=1}^{n} a_i c_i \left[k_i \sigma_i + \frac{Q_i}{2} + \frac{0.5}{2} \frac{\sigma_i^2}{Q_i} \left(1 - \exp\left(-\sqrt{2} \frac{Q_i}{\sigma_i} \right) \right) \exp\left(-\sqrt{2}\, k_i \right) \right]$$

subject to

$$\text{(I.2)} \qquad \sum_{i=1}^{n} \frac{0.5}{\sqrt{2}} \frac{Z_i D_i \sigma_i}{Q_i} \left[\left(1 - \exp\left(-\sqrt{2} \frac{Q_i}{\sigma_i} \right) \right) \right] \exp\left(-\sqrt{2} k_i \right) \leqslant \eta$$

where Z_i represents item essentiality. The Method of Lagrange yields

$$\text{(I.3)} \qquad k_i = -\frac{1}{\sqrt{2}} \ln \left[\frac{Q_i a_i c_i}{0.5 \left(1 - \exp\left(-\sqrt{2} \frac{Q_i}{\sigma_i} \right) \right) \left(\frac{a_i c_i \sigma_i}{\sqrt{2}} - \lambda Z_i D_i \right)} \right],$$

$$\text{(I.4)} \qquad \sum_{i=1}^{n} \frac{Z_i D_i \sigma_i a_i c_i}{\left(\frac{a_i c_i \sigma_i}{\sqrt{2}} - \lambda Z_i D_i \right)} = \sqrt{2}\eta, \text{ and}$$

$$\text{(I.5)} \qquad 0 = -\frac{A_i D_i}{Q_i^2} + \frac{a_i c_i}{2} - \frac{a_i c_i \sigma_i}{\sqrt{2} \left(1 - \exp\left(-\sqrt{2} \frac{Q_i}{\sigma_i} \right) \right)} \left[\frac{\left(1 - \exp\left(-\sqrt{2} \frac{Q_i}{\sigma_i} \right) \right)}{Q_i} - \frac{\sqrt{2}}{\sigma_i} \exp\left(-\sqrt{2} \frac{Q_i}{\sigma_i} \right) \right].$$

For reasons which will soon become evident, we will not solve (I.5) for Q_i until later.

MODEL II: Stockout penalty independent of duration of stockout; holding cost applied to inventory position.

We wish to minimize

$$\sum_{i=1}^{n} \frac{A_i D_i}{Q_i} + \sum_{i=1}^{n} a_i c_i \left(\mu_i + k_i \sigma_i + \frac{Q_i}{2}\right) \tag{II.1}$$

subject to

$$\sum_{i=1}^{n} \frac{0.5}{\sqrt{2}} \frac{Z_i D_i \sigma_i}{Q_i} \left(1 - \exp\left(-\sqrt{2}\frac{Q_i}{\sigma_i}\right)\right) \exp\left(-\sqrt{2} k_i\right) \leq \eta. \tag{II.2}$$

The Method of Lagrange yields

$$k_i = -\frac{1}{\sqrt{2}} \ln\left[\frac{Q_i a_i c_i}{0.5\ (-\lambda)\ Z_i D_i \left(1 - \exp\left(-\sqrt{2}\frac{Q_i}{\sigma_i}\right)\right)}\right], \tag{II.3}$$

$$-\lambda = \frac{1}{\sqrt{2}\,\eta} \sum_{i=1}^{n} \sigma_i a_i c_i, \text{ and} \tag{II.4}$$

$$0 = -\frac{A_i D_i}{Q_i^2} + \frac{a_i c_i}{2} - \frac{a_i c_i \sigma_i}{\sqrt{2}\left(1 - \exp\left(-\sqrt{2}\frac{Q_i}{\sigma_i}\right)\right)} \left[\frac{\left(1 - \exp\left(-\sqrt{2}\frac{Q_i}{\sigma_i}\right)\right)}{Q_i} - \frac{\sqrt{2}}{\sigma_i} \exp\left(-\sqrt{2}\frac{Q_i}{\sigma_i}\right)\right]. \tag{II.5}$$

MODEL III: Stockout penalty depends on duration of stockout; holding cost applied to on-hand inventory.

We wish to minimize

$$\sum_{i=1}^{n} \frac{A_i D_i}{Q_i} + \sum_{i=1}^{n} a_i c_i \left[k_i \sigma_i + \frac{Q_i}{2} + \frac{0.5}{2} \frac{\sigma_t^2}{Q_i} \left(1 - \exp\left(-\sqrt{2}\frac{Q_i}{\sigma_i}\right)\right) \exp\left(-\sqrt{2}\, k_i\right)\right] \tag{III.1}$$

subject to

$$\sum_{i=1}^{n} \frac{0.5}{2} \frac{Z_i \sigma_i^2}{Q_i} \left(1 - \exp\left(-\sqrt{2}\frac{Q_i}{\sigma_i}\right)\right) \exp\left(-\sqrt{2}\, k_i\right) \leq \beta. \tag{III.2}$$

The Method of Lagrange yields

$$k_i = -\frac{1}{\sqrt{2}} \ln\left[\frac{\sqrt{2}\, Q_i a_i c_i}{0.5\sigma_i \left(1 - \exp\left(-\sqrt{2}\frac{Q_i}{\sigma_i}\right)\right)(a_i c_i - \lambda Z_i)}\right], \tag{III.3}$$

$$\sum_{i=1}^{n} \frac{Z_i a_i c_i \sigma_i}{(a_i c_i - \lambda Z_i)} = \sqrt{2}\,\beta, \text{ and} \tag{III.4}$$

$$0 = -\frac{A_i D_i}{Q_i^2} + \frac{a_i c_i}{2} - \frac{a_i c_i \sigma_i}{\sqrt{2}\left(1 - \exp\left(-\sqrt{2}\frac{Q_i}{\sigma_i}\right)\right)} \left[\frac{\left(1 - \exp\left(-\sqrt{2}\frac{Q_i}{\sigma_i}\right)\right)}{Q_i} - \frac{\sqrt{2}}{\sigma_i} \exp\left(-\sqrt{2}\frac{Q_i}{\sigma_i}\right)\right]. \tag{III.5}$$

MODEL IV: Stockout penalty depends on duration of stockout; holding cost applied to inventory position.

We wish to minimize

$$\sum_{i=1}^{n} \frac{A_i D_i}{Q_i} + \sum_{i=1}^{n} a_i c_i \left(\mu_i + k_i \sigma_i + \frac{Q_i}{2}\right) \tag{IV.1}$$

subject to

$$\sum_{i=1}^{n} \frac{0.5}{2} \frac{Z_i \sigma_i^2}{Q_i} \left(1 - \exp\left(-\sqrt{2}\frac{Q_i}{\sigma_i}\right)\right) \exp\left(-\sqrt{2}\, k_i\right) \leq \beta. \tag{IV.2}$$

The Method of Lagrange yields

$$k_i = -\frac{1}{\sqrt{2}} \ln\left[\frac{\sqrt{2}\, Q_i a_i c_i}{0.5(-\lambda) Z_i \sigma_i \left(1 - \exp\left(-\sqrt{2}\frac{Q_i}{\sigma_i}\right)\right)}\right], \tag{IV.3}$$

$$-\lambda = \sum_{i=1}^{n} \frac{\sigma_i a_i c_i}{\sqrt{2}\,\beta}, \text{ and} \tag{IV.4}$$

$$0 = -\frac{A_i D_i}{Q_i^2} + \frac{a_i c_i}{2} - \frac{a_i c_i \sigma_i}{\sqrt{2}\left(1 - \exp\left(-\sqrt{2}\frac{Q_i}{\sigma_i}\right)\right)} \left[\frac{\left(1 - \exp\left(-\sqrt{2}\frac{Q_i}{\sigma_i}\right)\right)}{Q_i} - \frac{\sqrt{2}}{\sigma_i} \exp\left(-\sqrt{2}\frac{Q_i}{\sigma_i}\right)\right]. \tag{IV.5}$$

Notice that in all four models Q_i is the same (Eqs. (I.5), (II.5), (III.5), and (IV.5)). The quantity Q_i is completely independent of the backorder constraint regardless of whether or not the constraint is time-weighted. We also note that we cannot solve explicitly for Q. Conceptually, there is no problem. The exact solution for Q can be obtained through the use of an iterative routine (assuming access to a computer). An alternate and perhaps a more suitable approach is to recognize that in most cases

$$\left(1 - \exp\left(-\sqrt{2}\frac{Q_i}{\sigma_i}\right)\right) \approx 1$$

and can therefore be ignored. Consequently,

$$0 = -\frac{A_i D_i}{Q_i^2} + \frac{a_i c_i}{2} - \frac{a_i c_i \sigma_i}{\sqrt{2}\, Q_i} \tag{13}$$

and

$$Q_i = \frac{\sigma_i}{\sqrt{2}} + \sqrt{\frac{2 A_i D_i}{a_i c_i} + \left(\frac{\sigma_i}{\sqrt{2}}\right)^2}.^{*} \tag{14}$$

Notice that as $\sigma_i \to 0$, Eq. (14) approaches the well known Wilson lot size formula.

DISCUSSION – MODEL APPLICATION

Holding Cost Considerations

We wish now to discuss the difference between applying holding cost to the on-hand inventory (Models I and III) and applying it to the inventory position (Models II and IV). In spite of tradition, we feel it is more logical to apply the holding cost to the inventory position (see Ref. [4] for an argument supporting this point of view). From a practical point of view, the difference in results is small; however, the inventory position models have two very desirable attributes. First, there is an explicit solution

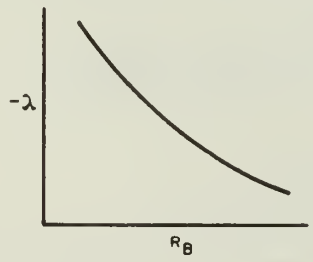
*It is easily shown that this value is an upper bound for the exact solution.

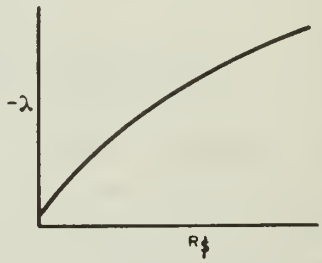
for λ, the Lagrangian, in these models—which is not the case when the holding cost is applied to the on-hand inventory. Second, in the inventory position models the safety level expressions are equivalent to those that would have been obtained had we somehow arrived at the buy quantities (Q_i's) and then independently minimized the cost of holding safety stock subject to the same backorder constraints used in Models I through IV. That is, under the inventory position models, the inventory manager who minimizes backorders subject to a safety level cost constraint would arrive at k_i's equal to the k_i's that he would have calculated had he minimized the total costs of ordering and holding inventory subject to a backorder constraint. This is especially important when the optimal Q_i's are not used: for example, when the policy is to purchase 3, 6, 9, or 12 months' worth of stock, or even when Q_i is determined by the Wilson lot size formula. If the same holding cost factor is applied to all items (i.e., $a_i = a$ for all i), the results obtained in models II and IV are equivalent to those determined when minimizing the safety level investment subject to a backorder constraint.

Negative k's

Although all four models are strongly biased against negative k's, as long as there is a heterogeneous inventory with some high cost, low demand items, there will be some items for which the models will specify negative k's. It could be argued that if even with a strong bias negative k's are obtained they should be accepted. We have no philosophical quarrel with this argument; however, in this paper, we insist that there be no negative safety levels. In order to accomplish this, we will set all negative k's equal to zero. Consequently, we will have more protection than the models call for and thus have fewer backorders, but a higher cost. The problem is to determine the λ that will allow, taking into account the nonnegativity of k, an acceptable number of backorders at a reasonable cost. Since we believe that Model IV is most appropriate for the Air Force's nonreparable items, we will use it to describe the methodology for determining λ.

Employing a simulation using actual data for about 5,000 Air Force items (for which seven years of history is available) we will select a range of values for $[-\lambda]$. Each value of $[-\lambda]$ will yield 5,000 k's—one for each item. The nonnegative k's will be accepted and the negative k's will be set equal to *zero*. We will then, using these k's which are now all nonnegative, obtain the average number of essentiality-weighted units in a backorder status divided by the number of essentiality-weighted units demanded annually (call this ratio R_B), and the cost of safety stock divided by the dollar value of annual demands (call this ratio $R_\$$). Subsequently, Figures 1 and 2 will be plotted as shown below:





Given Figures 1 and 2, the inventory managers can select either R_B or $R_\$$ (hopefully they will make the selection by considering both the cost and the effectiveness aspects) and then choose from the appropriate graph the value of $[-\lambda]$. The ability to choose on the basis of fund availability is extremely important since, in practice, the inventory manager is considerably more likely to be subject to a system constraint of dollars available for safety level investment (out of pocket cost) than to a back-

order constraint. Hadley and Whitin emphasize this point in their discussion of budget constraints. They conclude:

> "None of the inventory models discussed in operations research literature appears to provide an adequate treatment of budgetary constraint. In the first place, budgets are usually set in terms of actual dollars to be spent, whereas the costs included in inventory control models are often not the out-of-pocket cost type."*

An example of how the inventory manager (*IM*) might use these graphs (possibly to reduce "actual dollars to be spent") is as follows:

> Suppose the *IM* submits a budget request, for operating and safety stock, which allows 20 percent of the dollar value of annual demands for safety stock. If he receives the monies requested he can obtain from Figure 2, the $[-\lambda]$ that will yield an $R_{\$}$ of 20 percent. If he gets M dollars less than requested, and that M dollars represents 5 percent of annual dollar demands, he can set $R_{\$}$ equal to 15 percent (20% − 5%) and obtain the new value for $[-\lambda]$.

While this paper doesn't concern itself (other than the portion immediately preceding) with constrained optimization, we have applied the demand distribution described in this paper to models that do constrain the buy quantities (i.e., the Q's), the on-hand inventory, and the inventory position as well as the safety stock. None of these methods, however, is satisfactory when it comes to reducing actual dollars to be spent by very large amounts below computed requirements. All of these constraining models reduce dollars spent by reducing the reorder level (i.e., by reducing k) and/or by reducing the order quantities in some manner. We cannot, however, reduce the reorder level greatly without paying a very large price in terms of backorders. If most of our expenditures were for items with multi-year buy quantities, then we could temporarily reduce actual dollars to be spent by reducing these buys to a year or less. Unfortunately, by far the greatest majority of dollars are spent on items for which the buy quantity is on the order of three months worth of stock—which theoretically eliminates the possibility of obtaining a large reduction in actual dollars to be spent by reducing the buy quantity.

Backorder Costs Known

The reader will note that in the event the stockout cost (s_i) is known, the problem is simplified. The Q_i's remain unchanged and the k_i's of each of the models have a s_i in place of $[-\lambda]Z_i$ (see Eqs. (I.3), (II.3), (III.3), and (IV.3)). There is no $[-\lambda]$.

Final Note

The Air Force is planning to test all four models, nevertheless, at the present time, it favors (unofficially) Model IV. This model has also been brought to the attention of the Department of Defense by the Air Force for consideration during a series of recent joint services meetings designed to further standardize inventory policies and objectives among all DoD components.

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THE RELATIONSHIP BETWEEN DECISION VARIABLES AND PENALTY COST PARAMETER IN (Q, R) INVENTORY MODELS

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ABSTRACT

This paper is concerned with the optimum decision variables found using order quantity, reorder point (Q, R) inventory models. It examines whether the optimum variables (Q^* and R^*) are necessarily monotonic functions of the backorder cost parameter (or equivalently of the performance objective). For a general class of models it is proved that R^* must increase as the performance objective is raised, and an inequality condition is derived which governs how Q^* will change. Probability distributions of lead time demand are cited or found for which Q^* increases, Q^* decreases, and Q^* is independent of increases in performance objectives or backorder cost parameter.

INTRODUCTION

This paper is concerned with the optimum decision variables found by using order quantity, reorder point (Q, R) inventory models. It examines whether they are necessarily monotonic functions of the penalty (backorder) cost parameter (or equivalently of the performance objective). It attempts to make general statements about the direction of such monotonicity. Such information is of particular interest in contexts where the true cost of a backorder is difficult to estimate—and the performance objective is determined in large part by its impact on inventory levels.

It is easy to be misled by intuition. Our own empirical experience was with a (Q, R) model in which backorder cost is proportional to the length of time a backorder is outstanding, and lead time demand is normally distributed. For this case, the optimum reorder point (R^*) always increased as the penalty cost parameter (P) increased, while the optimum order quantity (Q^*) decreased. This is illustrated in Herron's [2] dimensionless graphs for the normal distribution. The conclusion can be proved analytically for deterministic demand and penalty proportional to time on backorder by solving directly for Q^*; e.g., see Hadley and Whitin's [1] solution.

On the other hand, Victor Presutti [3], found Q^* to be independent of the penalty cost parameter when he used a pseudo-exponential distribution for lead time demand.† That finding prompted this paper.

We initially assume penalty cost is proportional to the length of time a backorder is outstanding and begin by reviewing the relevant aspects of the appropriate (Q, R) model. Next we obtain some useful theoretical conclusions, including a proof that R^* must always increase when P does and an inequality condition which governs how Q^* will behave. We apply these results, and then in the last section extend them to other (Q, R) models.

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†$f(x) = \lambda_1 e^{-\lambda_2 x}$ for $x \geq 0$, and $f(x)$ defined elsewhere so that $\int_{-\infty}^{\infty} f(x)\,dx = 1$. Presutti's work is based on Lawrence Parker's approximation to the normal complementary cumulative.

The theoretical conclusions require the use of an approximation for average backorders (Eq. (2)). This approximation is commonly used in the actual computation of R^* and Q^* [1], [2], and does not negate the value of the results.

THE MODEL

The model is the basic steady state, continuous review (Q, R) model applicable when penalty cost is proportional to the length of time a backorder is outstanding. For the model to give strictly optimal answers in the general case, procurement lead times must be constant and demands must be for one unit at a time. However, the model is being used profitably even when these conditions do not hold. For ease of exposition we will assume that the lead time demand distribution is continuous. One good reference, particularly for providing the basis for Eqs. (1) and (2), is Hadley and Whitin [1].

Let

TC = average total cost per year;
H = holding cost per year per unit of stock on hand;
P = penalty cost per year per unit of stock on backorder;
A = administrative and/or setup cost;
λ = average yearly demand;
λ_L = expected demand during procurement lead time;
B = average amount on backorder at a random point in time;
$G(x)$ = probability that lead time demand equals or exceeds x;
L = lead time.

Then

$$TC(R,Q) = A(\lambda/Q) + H \cdot (R + Q/2 - \lambda_L + B) + P \cdot B = A \cdot (\lambda/Q) + H \cdot (R + Q/2 - \lambda_L) + P'B \tag{1}$$

$$P' = P + H$$

$$B = \frac{1}{Q}\int_R^\infty (x-R)[G(x) - G(x+Q)]\,dx.$$

For computational purposes, the following approximation is often used for B:

$$B = \frac{1}{Q}\int_R^\infty (x-R)G(x)\,dx. \tag{2}$$

The approximation is valid to the extent that the probability of demand exceeding $R+Q$ is small. We will utilize this expression for B.

THEORETICAL RESULTS

Let

Q_1, R_1 be the optimal decision variables for $P' = P_1$;

Q_2, R_2 be the optimal decision variables for $P' = P_2$;

$P_2 > P_1$;

B_i be expected average backorders for decision variables Q_i, R_i;

$$F(Q,R)=\frac{\partial B/\partial Q}{\partial B/\partial R}.$$

The following two theorems will be proved:

THEOREM 1: R^* (the optimal R) must increase as P increases.

THEOREM 2: The sign of $\partial F(Q,R)/\partial R$, if it is consistent, determines whether Q^* will also increase.

The key to these results is the corollary to Lemma 1 that the algebraic sign of Q_2-Q_1 is determined by the algebraic sign of $F(Q_2, R_2)-F(Q_1, R_1)$. Lemma 2 and its corollary demonstrate the intuitively "obvious": that expected backorders (B) must decrease in the optimum as P increases and, therefore, that it is not possible for both Q^* and R^* to decrease. Lemma 3 shows that $\partial F(Q, R)/\partial Q$ is always negative. Lemma 2 is used in proving Theorem 1, while Lemma 3 is used in proving Theorem 2.

LEMMA 1: The algebraic sign of $F(Q_2, R_2)-F(Q_1, R_1)$ is the same as the algebraic sign of Q_2-Q_1. (A zero value for one implies a zero value for the other.)

PROOF: Taking the partial derivatives of the total cost equation, Eq. (1), with respect to Q^* and R^*, and setting the derivatives to zero gives two conditions:

(3a) $$(\partial B/\partial Q^*)\cdot P'-\lambda\cdot A/(Q^*)^2+H/2=0;$$

(3b) $$(\partial B/\partial R^*)\cdot P'+H=0.$$

By algebraic manipulation (transposing and dividing one equation by the other), this gives

$$F(Q^*,R^*)=[\lambda\cdot A/(Q^*)^2-H/2]\div -H.$$

Hence

$$F(Q_2, R_2)-F(Q_1, R_1)=[\lambda\cdot A/(Q_2)^2-\lambda\cdot A/(Q_1)^2]\div -H.$$

If Q_2 exceeds Q_1(i.e., Q_2-Q_1 is positive), then both the numerator and denominator on the right hand side of Eq. (4) are negative; $F(Q_2, R_2)-F(Q_1, R_1)$ is then positive. By similar reasoning the rest of Lemma 1 is shown to hold.

COROLLARY: The algebraic sign of Q_2-Q_1 is the same as the algebraic sign of $F(Q_2, R_2)-F(Q_1, R_1)$. A zero value for one implies a zero value for the other.

LEMMA 2: The expected value of backorders (in the optimum) must decrease as P increases.

PROOF: To simplify the explanation let O_i stand for operating costs corresponding to decision variables (Q_i, R_i) so that Total Cost $=O_i+B_i\cdot P'$.

We use proof by contradiction. We initially assume $B_2>B_1$. By the optimality of Q_1, R_1 for $P'=P_1$,

$$O_2+B_2\cdot P_1>O_1+B_1\cdot P_1$$

Using $P_2>P_1$ and $B_2>B_1$, we find

$$O_2+B_2\cdot P_1+B_2\cdot(P_2-P_1)>O_1+B_1\cdot P_1+B_1\cdot(P_2-P_1).$$

Hence

$$O_2+B_2\cdot P_2>O_1+B_1\cdot P_2.$$

But this last inequality contradicts the optimality of Q_2, R_2 when $P'=P_2$.

COROLLARY: Both Q^* and R^* cannot decrease as P increases (proved by Lemma 2 and inspection of Eq. (2)).

LEMMA 3: $\partial F(Q, R)/\partial Q$ is always negative.

PROOF: Using Eq. (2), we get by calculus,

(5a) $$\frac{\partial B}{\partial Q}=-\frac{1}{Q^2}\int_R^\infty (x-R)G(x)dx$$

and

(5b) $$\frac{\partial B}{\partial R}=-\frac{1}{Q}\int_R^\infty G(x)dx.$$

Hence,

(6) $$F(Q, R)=\frac{-\frac{1}{Q^2}\int_R^\infty (x-R)G(x)dx}{-\frac{1}{Q}\int_R^\infty G(x)dx}=\frac{\frac{1}{Q}\int_R^\infty (x-R)G(x)dx}{\int_R^\infty G(x)dx}=\frac{B}{\int_R^\infty G(x)dx}.$$

The denominator does not depend on Q. Clearly, the numerator decreases as Q increases.

THEOREM 1: R^* increases as P increases.

PROOF: Using Eq. 6, we have

(7a) $$F(Q_2, R_2)=B_2\div\int_{R_2}^\infty G(x)dx$$

and

(7b) $$F(Q_1, R_1)=B_1\div\int_{R_1}^\infty G(x)dx.$$

Suppose $R_2<R_1$. Then the denominator of (7a) will be larger than the denominator of (7b). By Lemma 2, $B_2<B_1$, so this means $F(Q_2, R_2)<F(Q_1, R_1)$. By the corollary to Lemma 1 this implies $Q_2<Q_1$. By the corollary to Lemma 2, however, it is not possible for $Q_2<Q_1$ if $R_2<R_1$ since $P_2>P_1$. Proof by contradiction.

THEOREM 2A: If $\partial F(Q, R)/\partial R$ is consistently negative, then Q^* must decrease as P increases.

Since $R_2>R_1$, and $\partial F(Q, R)/\partial R$ is given as negative, $F(Q_1, R_2)<F(Q_1, R_1)$. Now if $Q_2>Q_1$, then by Lemma 3 this means $F(Q_2, R_2)<F(Q_1, R_2)$. By combining the two inequalities, we get $F(Q_2, R_2)<F(Q_1, R_1)$. However, by the Corollary to Lemma 1, this implies $Q_1>Q_2$. Proof by contradiction.

THEOREM 2B: If $\partial F(Q, R)/\partial R$ is consistently positive, then Q^* increases with P.

The proof is analagous to that of Theorem 2A. Since $R_2>R_1$, and $\partial F(Q, R)/\partial R$ is given as positive, $F(Q_1, R_2)>F(Q_1, R_1)$. Now if $Q_2<Q_1$, then by Lemma 3 this means $F(Q_2, R_2)>F(Q_1, R_2)$. By combining the two inequalities we get $F(Q_2, R_2)>F(Q_1, R_1)$. But by the Corollary to Lemma 1 this implies $Q_1<Q_2$. Proof by contradiction.

THEOREM 2C: If $F(Q, R)$ is independent of R, then Q^* is independent of P. For this case $F(Q_1, R_2)=F(Q_1, R_1)$. If $Q_2<Q_1$ then by Lemma 3, $F(Q_2, R_2)>F(Q_1, R_1)$, which by the Corollary to Lemma 1 implies $Q_2>Q_1$. Similar reasoning shows that the assumption $Q_1<Q_2$ also leads to a contradiction.

Applications

There is a strong temptation to reason intuitively that Q^* must increase as penalty cost increases: higher reorder quantities reduce the probability of stock out; this advantage becomes more significant as the penalty cost parameter is increased, while the impact of Q on ordering and holding costs is unchanged. Why is this reasoning wrong?

The flaw is that when penalty cost increases, it becomes worthwhile to increase R^*, so that changing Q^* has less impact on backorders than before, and therefore may have less impact on penalty

costs (the product of backorders and the penalty cost parameter). The decisive factor relates to the relative effectiveness in reducing backorders of augmenting Q^* vs augmenting R^*. As R increases, does the relative effectiveness of augmenting Q increase? Only then will Q^* increase as P increases.

It is difficult to apply Theorem 2 analytically to most probability distributions, i.e., to determine analytically the sign (+, −, 0 or mixed) of $\partial F(Q, R)\partial R$. No attempt was made to carry on an extensive computer aided mapping of sign vs probability distribution.

We did attempt to find a case when Q^* would increase as penalty cost increases. Recall that for the normal distribution Q^* decreases and that for exponential type distributions Q^* is unchanged.

Let the Weibull distribution be defined:

$$f(x) = \alpha\lambda x^{\alpha-1} \exp(-\lambda x^{\alpha});\ x \geq 0,\ \lambda\alpha \geq 0.$$

Then for $\alpha = 1/2$, we have analytically shown that $\partial F(Q, R)/\partial R$ is positive for $R \geq 0$, i.e., this is a case where Q^* increases with R. Empirical work indicates that this conclusion about the Weibull distribution also holds true when the precise expression for average backorders (Eq. (1)) is used.†

For $\alpha = 1$ the Weibull reduces to the exponential. It is simple to show $\partial F(Q, R)/\partial R = 0$ for the exponential regardless of which expression for B is used.

As a matter of interest, the Weibull with $\alpha = 1/2$ was originally examined because of an inference made from an analogy between determination of the sign of $\partial F(Q, R)/\partial R$, and the sign of the derivative of the failure rate function of reliability theory, the latter being a standard problem.

EXTENSIONS

The general linear form for penalty cost is $\bar{P} + P \cdot t$ where t is the time on backorder. This translates into total expected penalty cost $= P \cdot B + \bar{P} \cdot \bar{B}$ or, in words (cost per unit backordered per year on backorder) × (average amount on backorder) + (fixed cost per backorder occasion) × (number of demands backordered each year). That part of the total cost equation, Eq. (1), dependent on B or $\bar{B}$ becomes $P' \cdot B + \bar{P} \cdot \bar{B}$ where $P' = P + H$.

The theory derived in this paper still holds provided that as the penalty cost parameters are increased, $\bar{P}$ remains equal to $k \cdot P'$ for constant k (or $\bar{P} = 0$, the special case already considered). If the cost item $H \cdot B$ is left out of the total cost equation, Eq. (1), the condition is simply that $\bar{P} = k \cdot P$, which is more natural. $H \cdot B$ is normally very small, and in fact does not appear as a cost if holding cost depends on inventory position (including net due-in) rather than on hand.

An approximate expression must be used for $\bar{B}$ as well as B.††

Let

$$\bar{B} = \frac{\lambda}{Q}\int_R^\infty G(X)dx$$

$$B_T = k \cdot \bar{B} + B.$$

For $F_T(Q, R)$, defined in terms of $\partial B_T/\partial Q$ and $\partial B_T/\partial R$, we have

$$F_T(Q, R) = \frac{B_T}{\int_R^\infty G(x)dx + k \cdot \lambda \cdot G(R)}.$$

---

†Miss Chung-Mei Ho did much of the analytical investigation on the application of Theorem 2, and Mr. Robert Deemer did the empirical work for the Weibull.

††Hadley and Whitin, chapter 4 has a derivation.

By inspection, all three lemmas proved hold with the substitutions of B_T for B, $P' \cdot B_T$ for $P'B$, and $F_T(Q, R)$ for $F(Q, R)$. Therefore, the two theorems hold with these substitutions.

Using the precise expression for B causes difficulties even when $\bar{P}$ is 0. Then

$$F(Q,R) = \frac{B - \int_{R+Q}^{\infty} [y-(R+Q)]g(y)dy}{\int_{R}^{\infty} [G(x)-G(x+Q)]dx}, \text{ where } g(y) = -\frac{dG(y)}{dy}.$$

It is no longer apparent that Lemma 3 must always hold, or that in the proof of Theorem 1 that the numeraton $F(Q_2, R_2) <$ numerator of $F(Q_1, R_1)$ if $Q_2 > Q_1$.

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* * *

NEWS AND MEMORANDA

PROFESSIONAL DESIGNATION IN LOGISTICS MANAGEMENT

The School of Systems and Logistics, Air Force Institute of Technology, in cooperation with the Society of Logistics Engineers (SOLE), has announced the beginning of a program of professional designation for qualified military and Civil Service logistics managers of the Department of Defense. Those who meet the criteria may be awarded an appropriate certificate of "Professional Designation in Logistics Management." Certification is not restricted only to members of the Society.

The purpose of the program is to recognize the efforts of the individual to increase his educational background in the broad world of logistics. It recognizes that true logistics management capability must encompass more than expertise in one of the several disciplines which comprise the logistics spectrum. The idea is that logistics is inherently an integrating process of these several disciplines and that the effective logistics manager will be one who has knowledge in, and understands the viewpoint of, the various sub-functions which the separate disciplines represent. The professional stature of the functional specialists is recognized, but this program is aimed at the need for a portion of these managers to become logistics generalists to perform the essential function of integrating the efforts of many functional specialists.

Qualification for certification rests upon the types and forms of education and experience which contribute to the overall concept of integrative management in logistics. The courses of study outlined in the qualification listing were tailored with this concept in mind. The objective is to assure that those who qualify have progressed from one of the functional specialties to a broader realization of logistics management needs through exposure to education in related and supporting disciplines and functional areas. Thus, those who qualify for certification will have demonstrated a high degree of initiative in acquiring a broad educational and experience background. They will be considered to have demonstrated their competency for advancement to higher levels of responsibility in logistics management where their broadened point of view would aid decision-making of more cost-effective nature.

The certification program provides for two levels of certificate award. The first is "Professional Designation in Logistics Management" and the second is "Professional Designation in Logistics Management—Advanced." The qualification criteria includes courses in five categories as indicated:

		Certificate	*Advanced*
Category A	Logistics Integration	2	2
Category B1	Materiel Acquisition	1	2
Category B2	Materiel Distribution	2	2
Category B3	Materiel Maintenance	1	2
Category C	Management Techniques	2	2
	Total Required	8	10

Some allowance is made in the program for waiving certain educational course requirements on the basis of field experience. Such waiver, for a maximum of two courses, must be substantiated by the applicant. The determination of whether or not to accept the experience equivalency is left to the discretion of the certification committee of the School of Systems and Logistics. Applicants are urged

to explain their request for equivalency and waiver by an adequate explanation of the experience (where, for how long, in what job, at what level, and so on).

It is also likely that many will have taken higher level educational courses than those listed in the certificate categories. These may be accepted in lieu of the described courses if the applicant includes proof of completion along with a brief outline of material covered.

Certificates will be approved by the School of Systems and Logistics and the SOLE National Education Committee through a certification board process. Applicants, who need not be members of SOLE but who must be members of DOD, are encouraged to submit their requests for certification to the School of Systems and Logistics (AFIT–SL), W–PAFB, Ohio 45433. Each request must be accompanied by the necessary back-up data.

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